

## Ondes de surface

MEC567 : SCIENCES DE L'EAU ET ENVIRONNEMENT

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## Introduction

### 1. Génération des ondes de surface

Le modèle des équations d'Euler irrotationnelles et linéaires décrit les petites oscillations de deux couches fluides superposées. Il rend compte de la génération des vagues et de leur dispersion.

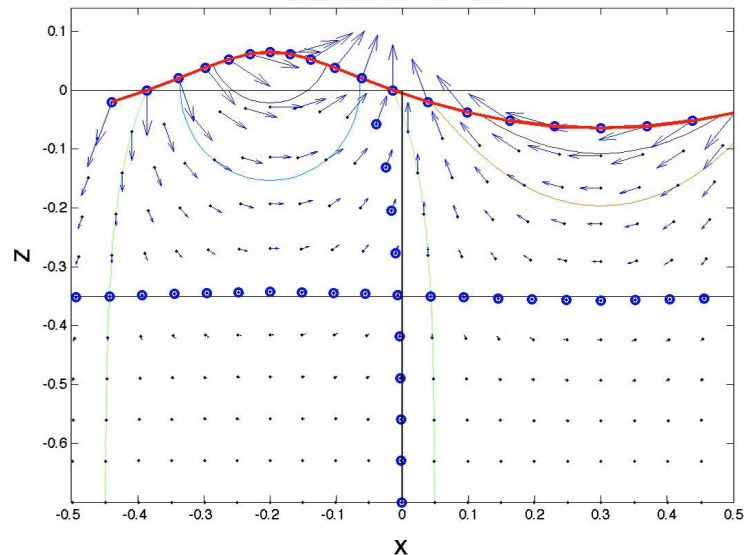
### 2. Dispersion de la houle

Les crêtes d'une onde monochromatique se déplacent à la vitesse de phase, toujours plus grande que la vitesse de groupe, et les trajectoires des particules sont des ellipses.

### 3. Problèmes aux conditions initiales

Une condition initiale quelconque génère des paquets d'ondes à droite et à gauche, qui se dispersent en se déplaçant à leurs vitesses de groupe respectives.

pression,  $h = 0.7$

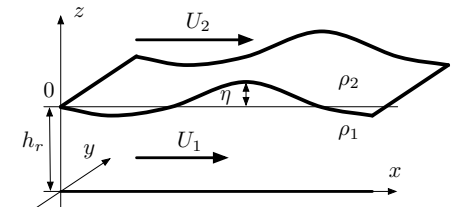


## Équations d'Euler incompressibles

$$\begin{aligned} \operatorname{div} \underline{U}_2 &= 0 & \text{et} & & \frac{\partial \underline{U}_2}{\partial t} + \underline{U}_2 \cdot \operatorname{grad} \underline{U}_2 &= -\frac{1}{\rho_2} \operatorname{grad} p_2 - g \underline{e}_z, \\ \operatorname{div} \underline{U}_1 &= 0 & \text{et} & & \frac{\partial \underline{U}_1}{\partial t} + \underline{U}_1 \cdot \operatorname{grad} \underline{U}_1 &= -\frac{1}{\rho_1} \operatorname{grad} p_1 - g \underline{e}_z \end{aligned}$$

$$\lim_{z \rightarrow +\infty} \underline{U}_2 = U_2 \underline{e}_x$$

$$\underline{U}_1 \cdot \underline{e}_z = 0, \quad z = -h_r$$



## Conditions aux limites à l'interface

$$\frac{\partial \eta}{\partial t} + \underline{U}_1 \cdot \operatorname{grad} \eta = w_1, \quad p_1 = p_2, \quad \frac{\partial \eta}{\partial t} + \underline{U}_2 \cdot \operatorname{grad} \eta = w_2$$

Hypothèse irrationnelle :  $\text{rot } \underline{U}_1 = \text{rot } \underline{U}_2 = \underline{0}$

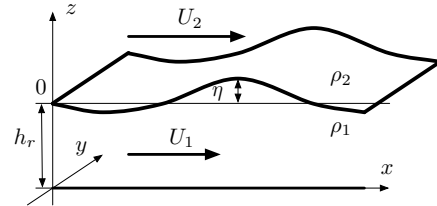
$$\underline{U}_2 = \text{grad } (U_2 x + \phi_2) \quad \text{et} \quad \underline{U}_1 = \text{grad } (U_1 x + \phi_1)$$

$$\Delta \phi_2 = 0 \quad \text{et} \quad \text{grad} \left[ \frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} + \frac{1}{2} (\text{grad } \phi_2)^2 + \frac{\rho_2}{\rho_1} + g z \right] = 0$$

$$\Delta \phi_1 = 0 \quad \text{et} \quad \text{grad} \left[ \frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} + \frac{1}{2} (\text{grad } \phi_1)^2 + \frac{\rho_1}{\rho_1} + g z \right] = 0$$

$$\lim_{z \rightarrow +\infty} \text{grad } \phi_2 = \underline{0}$$

$$\frac{\partial \phi_1}{\partial z} = 0 \quad \text{en } z = -h_r$$



$$\left( \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} \right) \eta + \text{grad } \phi_i \cdot \text{grad } \eta = \frac{\partial \phi_i}{\partial z}, \quad i = \begin{cases} 1 \\ 2 \end{cases} \quad \text{et} \quad \rho_1 = \rho_2$$

Petites oscillations :  $\eta$  "petit"

$$f[x, y, \eta(x, y, t), t] = f(x, y, 0, t) [1 + O(\eta)]$$

$$\Delta \phi_2 = 0$$

$$\lim_{z \rightarrow +\infty} \text{grad } \phi_2 = \underline{0}$$

Condition aux limites en  $z = 0$

$$\left( \frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x} \right) \eta = \frac{\partial \phi_1}{\partial z} \quad \text{et} \quad \left( \frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x} \right) \eta = \frac{\partial \phi_2}{\partial z}$$

$$\rho_1 \left[ \left( \frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x} \right) \phi_1 + g \eta \right] = \rho_2 \left[ \left( \frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x} \right) \phi_2 + g \eta \right]$$

$$\Delta \phi_1 = 0$$

$$\frac{\partial \phi_1}{\partial z} = 0 \quad \text{en } z = -h_r$$

Solutions complexes avec  $s = \sigma - i\omega$  :

$$(\phi_1, \eta, \phi_2) = [\Phi_1(z), \eta_m, \Phi_2(z)] e^{ik_x x + ik_y y + st}$$

En notant  $k^2 = k_x^2 + k_y^2$ , le système s'écrit :

$$\begin{aligned} \Phi_1'' - k^2 \Phi_1 = 0 \quad \text{et} \quad \Phi_2'' - k^2 \Phi_2 = 0 \quad \text{avec} \\ (s + i k_x U_1) \eta_m = \Phi_1'(0) \quad \text{et} \quad (s + i k_x U_2) \eta_m = \Phi_2'(0) \\ \rho_1 [(s + i k_x U_1) \Phi_1(0) + g \eta_m] = \rho_2 [(s + i k_x U_2) \Phi_2(0) + g \eta_m] \\ \Phi_1'(-h_r) = 0 \quad \text{et} \quad \lim_{z \rightarrow \infty} \Phi_2'(z) = 0 \end{aligned}$$

$$\text{D'où : } \Phi_1(z) = \Phi_{1m} \cosh(kz + kh_r) \quad \text{et} \quad \Phi_2(z) = \Phi_{2m} e^{-kz}$$

On en déduit la relation de dispersion :

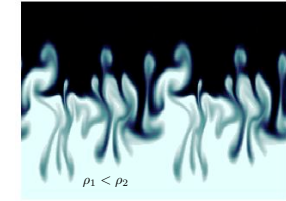
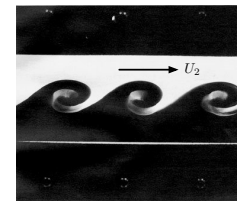
$$\rho_1 \left[ gk + \frac{(s + i k_x U_1)^2}{\tanh(kh_r)} \right] = \rho_2 [gk - (s + i k_x U_2)^2]$$

Cas de la profondeur infinie  $kh_r \rightarrow \infty$

$$\rho_1 [gk + (s + i k_x U_1)^2] = \rho_2 [gk - (s + i k_x U_2)^2]$$

Condition nécessaire et suffisante pour l'instabilité

$$g \sqrt{k_x^2 + k_y^2} (\rho_1^2 - \rho_2^2) < k_x^2 \rho_1 \rho_2 (U_1 - U_2)^2$$



Si  $\rho_1 > \rho_2$

Instable pour  $U_1 \neq U_2$   
et  $|U_1 - U_2|$  suffisamment fort

Si  $U_1 = U_2 = 0$

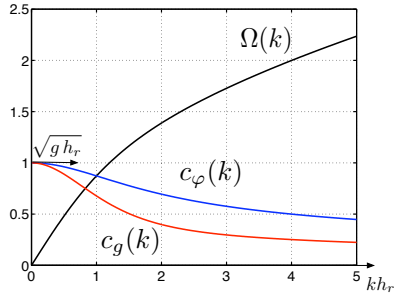
$$s^2 = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} gk$$

Cas d'une profondeur  $h_r$  quelconque avec  $\rho_1 \gg \rho_2$

$$(s + i k_x U_1)^2 + g k \tanh(k h_r) = 0$$

Dans le repère mobile de vitesse  $U_1$

$$\omega = \Omega(k) = \sqrt{g k \tanh(k h_r)}$$



Vitesse de phase

$$c_\varphi(k) = \frac{\Omega(k)}{k}$$

Vitesse de groupe

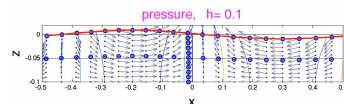
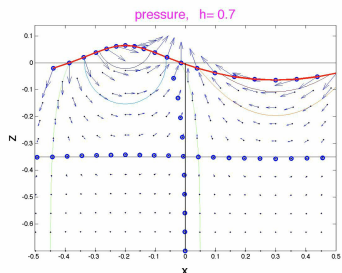
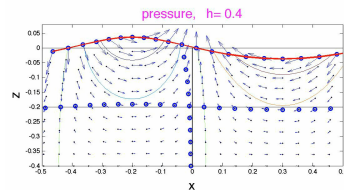
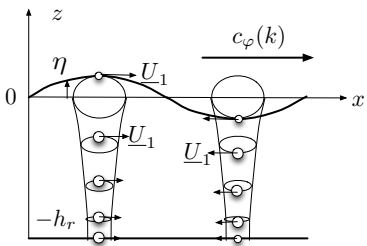
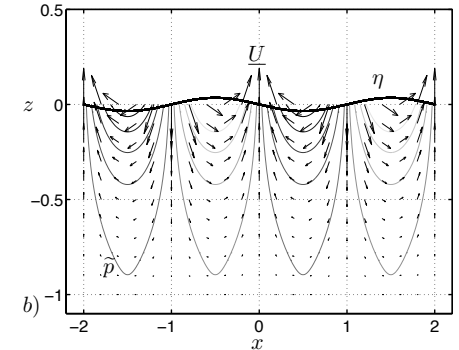
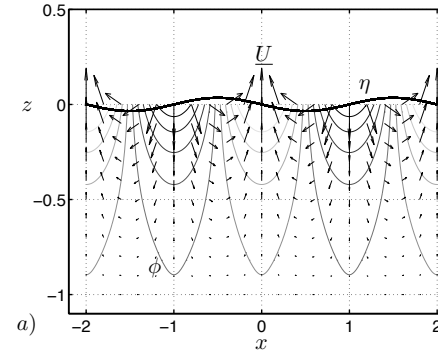
$$c_g(k) = \Omega'(k) = c_\varphi(k) \left[ \frac{1}{2} + \frac{k h_r}{\sinh(2 k h_r)} \right]$$

Onde monochromatique avec  $\underline{U} = \text{grad } \phi$

$$\eta = \eta_m \cos(k_x x + k_y y - \omega t)$$

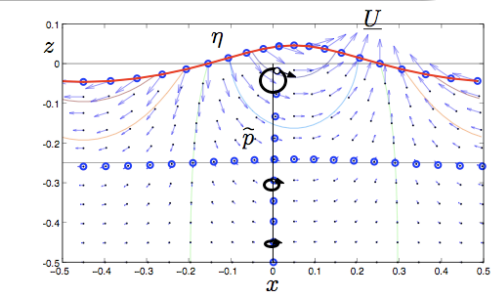
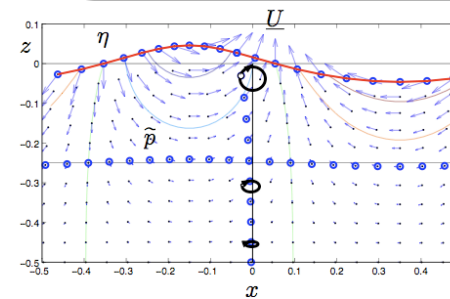
$$\phi = \frac{g \eta_m}{\omega} \sin(k_x x + k_y y - \omega t) \cosh[k(z + h_r)] / \cosh(k h_r)$$

$$\tilde{p} = \rho g \eta_m \cos(k_x x + k_y y - \omega t) \cosh[k(z + h_r)] / \cosh(k h_r)$$



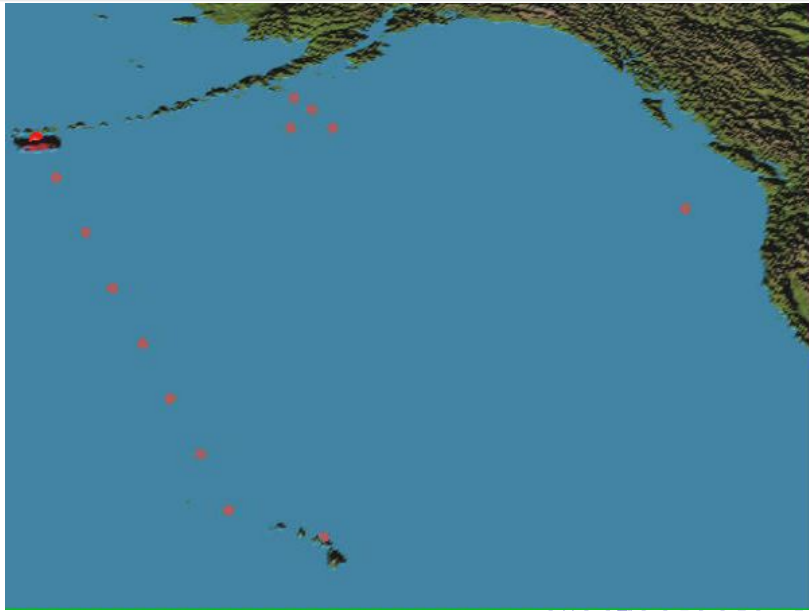
$$u = (g \eta_m / \omega) k_x \cosh[k(z + h_r)] \cos(k_x x + k_y y - \omega t)$$

$$w = (g \eta_m / \omega) k \sinh[k(z + h_r)] \sin(k_x x + k_y y - \omega t)$$



$$x(x_0, z_0; t) = x_0 - \frac{g \eta_m}{\omega^2} k_x \frac{\cosh[k(z_0 + h_r)]}{\cosh(k h_r)} \sin(k_x x_0 - \omega t)$$

$$z(x_0, z_0; t) = z_0 + \frac{g \eta_m}{\omega^2} k \frac{\sinh[k(z_0 + h_r)]}{\cosh(k h_r)} \cos(k_x x_0 - \omega t)$$



Condition initiale (vérifiant  $\Delta\phi_0 = 0$  et  $\frac{\partial}{\partial z}\phi_0 = 0$  en  $z = -h_r$ )

$$\eta_0(x) = \int_{\mathbb{R}} \hat{\eta}_0(k_x) e^{i k_x x} dk_x$$

$$\phi_0(x, z) = \int_{\mathbb{R}} \hat{\phi}_0(k_x) \cosh[k(z + h_r)] e^{i k_x x} dk_x$$

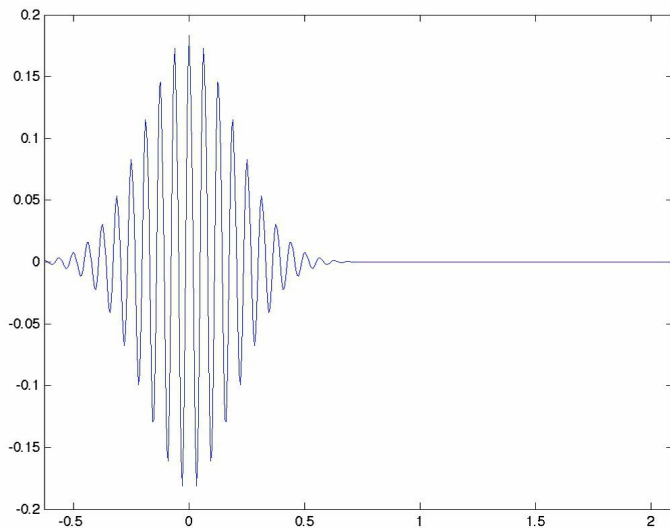
Ondes à Gauche / Droite avec  $\Omega(k) = \sqrt{g k \tanh(k h_r)}$

$$\hat{\eta}_0(k_x) = \hat{\eta}_{G0}(k_x) + \hat{\eta}_{D0}(k_x) \quad \text{et} \quad \hat{\phi}_0(k_x) = \hat{\phi}_{G0}(k_x) + \hat{\phi}_{D0}(k_x)$$

Cas d'un paquet d'ondes à droites ( $\hat{\eta}_{G0} = 0$ )

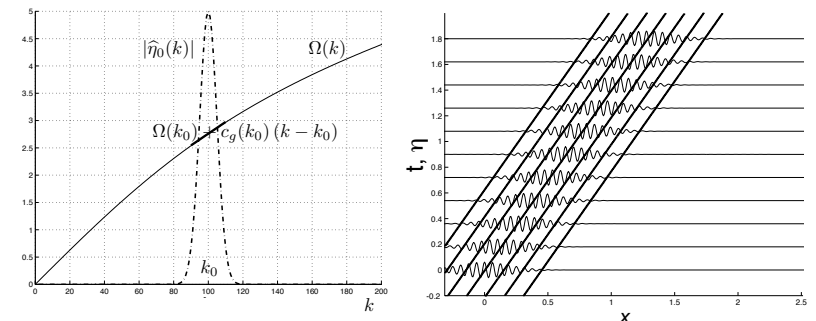
$$\eta(x, t) = \int_{\mathbb{R}^+} \hat{\eta}_0(k_x) e^{i k_x x - i \Omega(k) t} dk_x + c.c.$$

$$\phi(x, z, t) = \int_{\mathbb{R}^+} \frac{g \hat{\eta}_0(k_x)}{i \Omega(k)} \frac{\cosh[k(z + h_r)]}{\cosh(k h_r)} e^{i k_x x - i \Omega(k) t} dk_x + c.c.$$



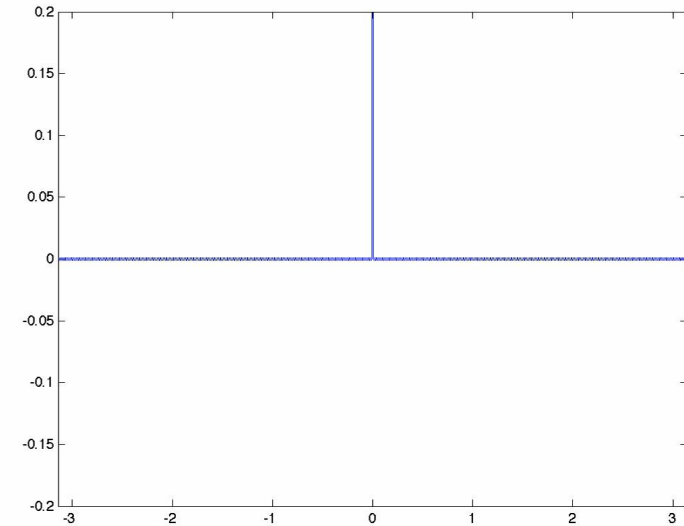
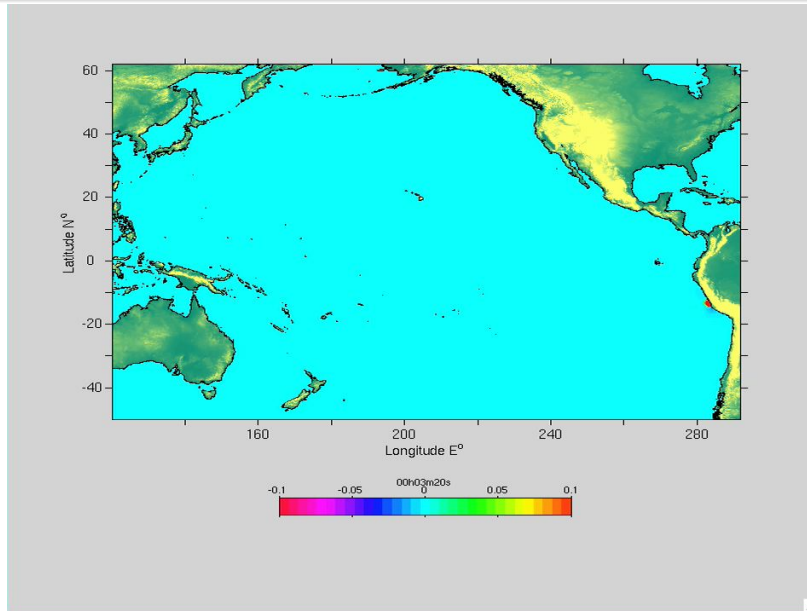
Condition initiale  $\eta_0(x) = 2 \eta_p \cos(k_0 x) E(x)$  avec

$$\hat{E}(q) = \exp\left(\frac{-q^2}{2\chi^2}\right) \iff E(x) = \chi \sqrt{2\pi} \exp\left(-\frac{\chi^2 x^2}{2}\right)$$



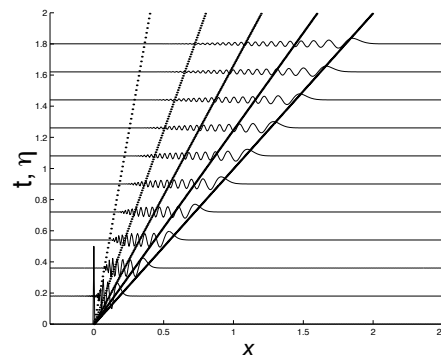
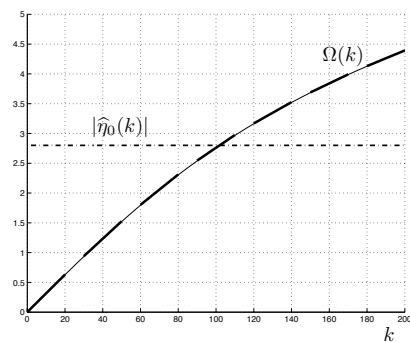
Paquet d'ondes :  $\Omega(k) = \Omega(k_0) + c_g(k_0)(k - k_0) + \dots$

$$\eta(x, t) = 2 \eta_p \cos[k_0 x - \Omega(k_0) t] E[x - c_g(k_0) t]$$



Condition initiale

$$\hat{\eta}_0(k) = \eta_m \iff \eta_0(x) = 2\pi \eta_m \delta(x)$$



Réponse impulsionnelle  $\Omega(k) = \sqrt{g k \tanh(k h_r)}$

$$\eta(x, t) = \eta_m \int_{R^+} e^{i k_x x - i \Omega(k) t} dk_x + c.c. \text{ avec } k = |k_x|$$

Comportement le long de la trajectoire  $x = c t$

$$I(t) = \eta(c t, t) = \eta_m \int_{R^+} e^{i [k_x c - \Omega(k)] t} dk_x + c.c.$$

Méthode de la phase stationnaire :  $I(t) = \int \exp[i \psi(k)] dk$

si  $\Psi$  est monotone :  $I(t)$  décroît exponentiellement  
si  $k_*$  tel que  $\Psi'(k_*) = 0$  :  $I(t) \sim \frac{1}{\sqrt{t}} G(k_*) e^{i \Psi(k_*) t}$

Paquet d'onde dispersé :  $\Psi(k_x) = k_x c - \Omega(k)$

$$\eta(x, t) \sim \frac{1}{\sqrt{t}} G \left[ k_c \left( \frac{x}{t} \right) \right] e^{i k_c \left( \frac{x}{t} \right) x - \Omega \left[ k_c \left( \frac{x}{t} \right) \right] t}$$

où  $k_* = k_c(c)$  est le nombre d'onde défini par  $c_g(k_*) = c$   
et  $G(k_*) = \eta_m \sqrt{\frac{2\pi}{|\Omega''(k_*)|}} \exp \left\{ -i \text{sign}[\Omega''(k_*)] \frac{\pi}{4} \right\}$