Ondes de surface

HYDRODYNAMIQUE DE L'ENVIRONNEMENT, O. THUAL

28 septembre 2011

Introduction

1. Génération des ondes de surface

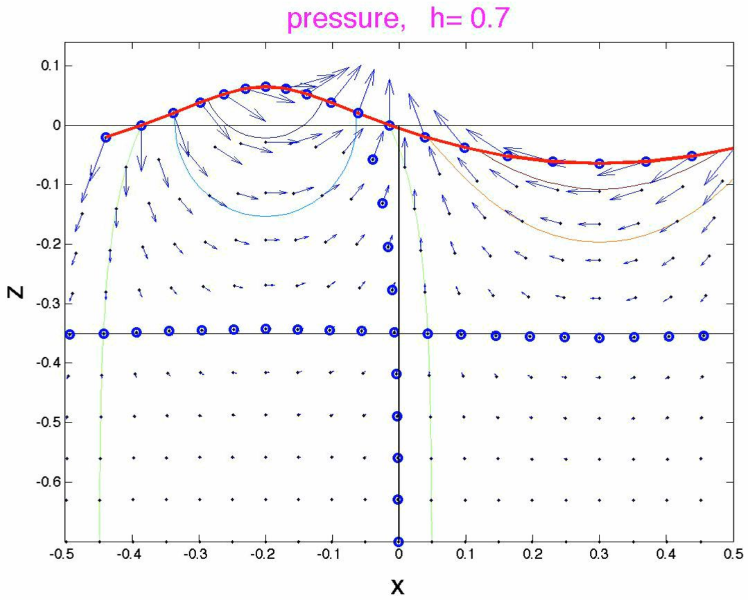
Le modèle des équations d'Euler irrotationnelles et linéaires décrit les petites oscillations de deux couches fluides superposées. Il rend compte de la génération des vagues et de leur dispersion.

2. Dispersion de la houle

Les crêtes d'une onde monochromatique se déplacent à la vitesse de phase, toujours plus grande que la vitesse de groupe, et les trajectoires des particules sont des ellipses.

3. Problèmes aux conditions initiales

Une condition initiale quelconque génère des paquets d'ondes à droite et à gauche, qui se dispersent en se déplaçant à leurs vitesses de groupe respectives.


pressure, h= 0.7
Z
X
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5

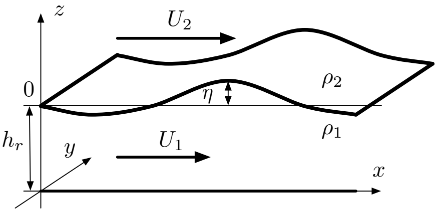
Équations d'Euler incompressibles

$$\text{div } \underline{U}_2 = 0 \qquad \text{et} \qquad \frac{\partial \underline{U}_2}{\partial t} + \underline{U}_2 \cdot \underline{\text{grad}}\ \underline{U}_2 = -\frac{1}{\rho_2} \underline{\text{grad}}\ p_2 - g\ \underline{e}_z\ ,$$

$$\text{div } \underline{U}_1 = 0 \qquad \text{et} \qquad \frac{\partial \underline{U}_1}{\partial t} + \underline{U}_1 \cdot \underline{\text{grad}}\ \underline{U}_1 = -\frac{1}{\rho_1} \underline{\text{grad}}\ p_1 - g\ \underline{e}_z$$

$$\lim_{z \to +\infty} \underline{U}_2 = U_2\ \underline{e}_x$$

$$\underline{U}_1 \cdot \underline{e}_z = 0\ , \quad z = -h_r$$



Conditions aux limites à l'interface $z = \eta(x, y, t)$

$$\frac{\partial \eta}{\partial t} + \underline{U}_1 \cdot \underline{\text{grad}}\ \eta = w_1\ , \qquad p_1 = p_2\ , \qquad \frac{\partial \eta}{\partial t} + \underline{U}_2 \cdot \underline{\text{grad}}\ \eta = w_2$$

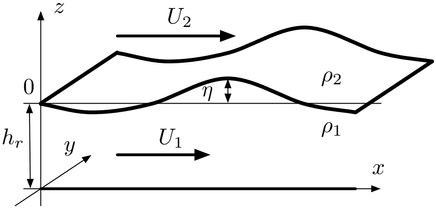
Hypothèse irrationnelle : $\underline{\text{rot}}\ \underline{U}_1 = \underline{\text{rot}}\ \underline{U}_2 = \underline{0}$

$$\underline{U}_i = \underline{\text{grad}}\ (U_i\ x + \phi_i) \implies \Delta\phi_i = 0\ , \quad i = \begin{cases} 2 \\ 1 \end{cases}$$

$$\underline{\text{grad}}\ \left[\frac{\partial \phi_i}{\partial t} + U_i \frac{\partial \phi_i}{\partial x} + \frac{1}{2}\left(\underline{\text{grad}}\ \phi_i\right)^2 + \frac{p_i}{\rho_i} + g\ z\right] = \underline{0}\ , \quad i = \begin{cases} 2 \\ 1 \end{cases}$$

$\lim_{z\to+\infty} \underline{\text{grad}}\ \phi_2 = \underline{0}$

$\frac{\partial \phi_1}{\partial z} = 0$ en $z = -h_r$



Conditions aux limites à l'interface $z = \eta(x, y, t)$

$$\left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x}\right)\eta + \underline{\text{grad}}\ \phi_i \cdot \underline{\text{grad}}\ \eta = \frac{\partial \phi_i}{\partial z}\ , \quad i = \begin{cases} 2 \\ 1 \end{cases} \qquad \text{et} \quad p_1 = p_2$$

Petites oscillations : η "petit"

$$f\left[x, y, \eta(x,y,t), t\right] = f(x,y,0,t)\ \left[1 + O(\eta)\right]$$

$$\Delta\phi_2 = 0$$

$$\lim_{z\to+\infty} \underline{\text{grad}}\ \phi_2 = \underline{0}$$

Condition aux limites en $z = 0$

$$\left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x}\right)\eta = \frac{\partial\phi_1}{\partial z} \qquad \text{et} \qquad \left(\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x}\right)\eta = \frac{\partial\phi_2}{\partial z}$$

$$\rho_1 \left[\left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x}\right)\phi_1 + g\,\eta\right] = \rho_2 \left[\left(\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x}\right)\phi_2 + g\,\eta\right]$$

$$\Delta\phi_1 = 0$$

$$\frac{\partial\phi_1}{\partial z} = 0 \ \text{ en } \quad z = -h_r$$

Solutions complexes avec $s = \sigma - i\,\omega$:

$$(\phi_1, \eta, \phi_2) = [\Phi_1(z), \eta_m, \Phi_2(z)]\, e^{ik_x\, x + ik_y\, y + st}$$

En notant $k^2 = k_x^2 + k_y^2$, le système s'écrit :

$$\begin{array}{rcl} \Phi_1'' - k^2\,\Phi_1 = 0 & \text{et} & \Phi_2'' - k^2\,\Phi_2 = 0 \quad \text{avec} \\ (s + i\,k_x\,U_1)\,\eta_m = \Phi_1'(0) & \text{et} & (s + i\,k_x\,U_2)\,\eta_m = \Phi_2'(0) \\ \rho_1\,[(s + i\,k_x\,U_1)\,\Phi_1(0) + g\,\eta_m] & = & \rho_2\,[(s + i\,k_x\,U_2)\,\Phi_2(0) + g\,\eta_m] \\ \Phi_1'(-h_r) = 0 & \text{et} & \lim_{z\to\infty} \Phi_2'(z) = 0 \end{array}$$

D'où : $\Phi_1(z) = \Phi_{1m}\cosh(k\,z + k\,h_r)$ et $\Phi_2(z) = \Phi_{2m}\,e^{-k\,z}$

On en déduit la relation de dispersion :

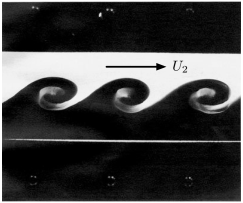
$$\rho_1 \left[g\,k + \frac{(s + i\,k_x\,U_1)^2}{\tanh(k\,h_r)} \right] = \rho_2 \left[g\,k - (s + i\,k_x\,U_2)^2 \right]$$

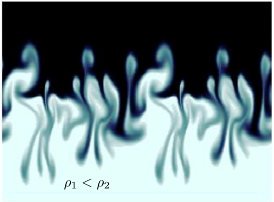
Cas de la profondeur infinie $kh_r \to \infty$

$$\rho_1 \left[g\, k + (s + i\, k_x\, U_1)^2\right] = \rho_2 \left[g\, k - (s + i\, k_x\, U_2)^2\right]$$

Condition nécessaire et suffisante pour l'instabilité

$$g\, \sqrt{k_x^2 + k_y^2}\, (\rho_1^2 - \rho_2^2) < k_x^2\, \rho_1\, \rho_2\, (U_1 - U_2)^2$$





Si $\rho_1 > \rho_2$

Instable pour $U_1 \neq U_2$
et $|U_1 - U_2|$ suffisamment fort

Si $U_1 = U_2 = 0$

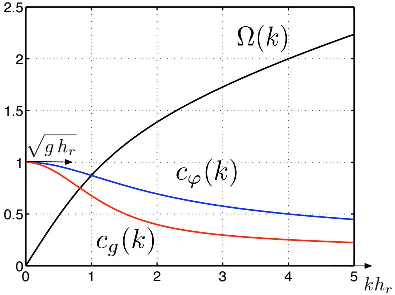
$$s^2 = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\, g\, k$$

Cas d'une profondeur h_r quelconque avec $\rho_1 \gg \rho_2$

$$(s + i\, k_x\, U_1)^2 + g\, k\, \tanh(k\, h_r) = 0$$

Dans le repère mobile de vitesse U_1

$$\omega = \Omega(k) = \sqrt{g\, k\, \tanh(k\, h_r)}$$



Vitesse de phase

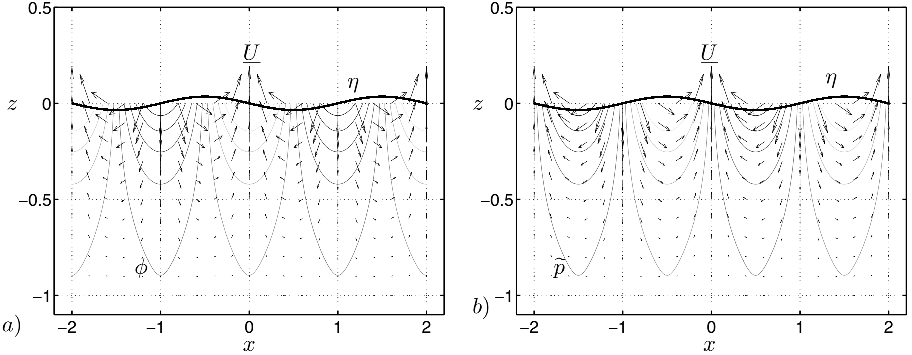
$$c_\varphi(k) = \frac{\Omega(k)}{k}$$

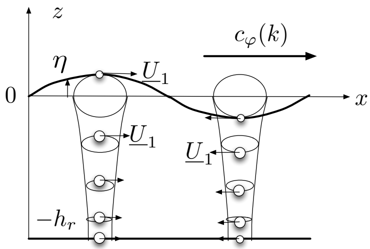
Vitesse de groupe

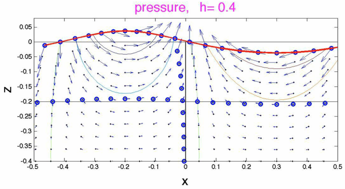
$$c_g(k) = \Omega'(k)$$
$$= c_\varphi(k)\ \left[\frac{1}{2} + \frac{k\, h_r}{\sinh(2\, k\, h_r)}\right]$$

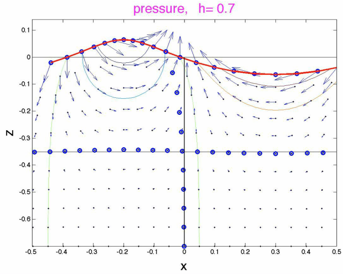
Onde monochromatique avec $\underline{U} = \underline{\text{grad}}\ \phi$

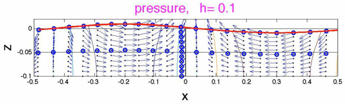
$$\begin{aligned}
\eta &= \eta_m \cos(k_x\, x + k_y\, y - \omega t) \\
\phi &= \frac{g\, \eta_m}{\omega} \sin(k_x\, x + k_y\, y - \omega t) \cosh[k\,(z + h_r)]/\cosh(k\, h_r) \\
\tilde{p} &= \rho\, g\, \eta_m \cos(k_x\, x + k_y\, y - \omega t) \cosh[k\,(z + h_r)]/\cosh(k\, h_r)
\end{aligned}$$



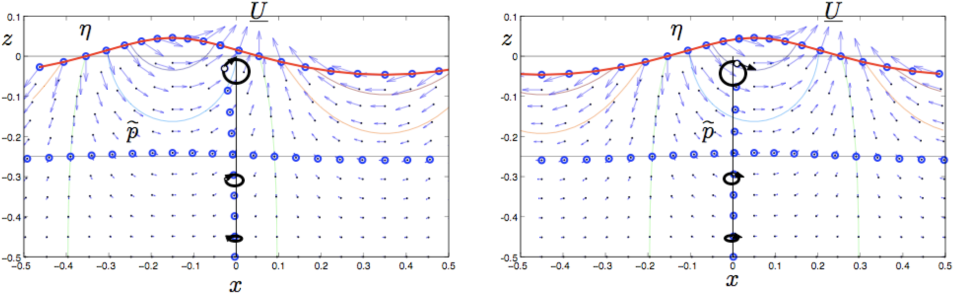

z
$c_\varphi(k)$
η
$\underline{U}_1$
0
x
$\underline{U}_1$
$\underline{U}_1$
$-h_r$


pressure, h= 0.4
z
x


pressure, h= 0.7
z
x


pressure, h= 0.1
z
x

$$
\begin{aligned}
u &= (g\,\eta_m/\omega)\ k_x\ \cosh[k(z+h_r)]\ \cos(k_x\,x + k_y\,y - \omega t)\\
w &= (g\,\eta_m/\omega)\ k\ \ \sinh[k(z+h_r)]\ \sin(k_x\,x + k_y\,y - \omega t)
\end{aligned}
$$



$$
\begin{aligned}
x(x_0,z_0;t) &= x_0 - \frac{g\ \eta_m}{\omega^2}\ k_x\ \frac{\cosh[k(z_0+h_r)]}{\cosh(k\,h_r)}\ \sin(k_x\,x_0 - \omega\,t)\\
z(x_0,z_0;t) &= z_0 + \frac{g\ \eta_m}{\omega^2}\ k\ \frac{\sinh[k(z_0+h_r)]}{\cosh(k\,h_r)}\ \cos(k_x\,x_0 - \omega\,t)\ ,
\end{aligned}
$$

Condition initiale (vérifiant $\Delta\phi_0 = 0$ et $\frac{\partial}{\partial z}\phi_0 = 0$ en $z = -h_r$)

$$\eta_0(x) = \int_{\mathbf{R}} \widehat{\eta}_0(k_x)\, e^{i\,k_x\,x}\; dk_x$$

$$\phi_0(x,z) = \int_{\mathbf{R}} \widehat{\phi}_0(k_x)\; \cosh[k\,(z+h_r)]\; e^{i\,k_x\,x}\; dk_x$$

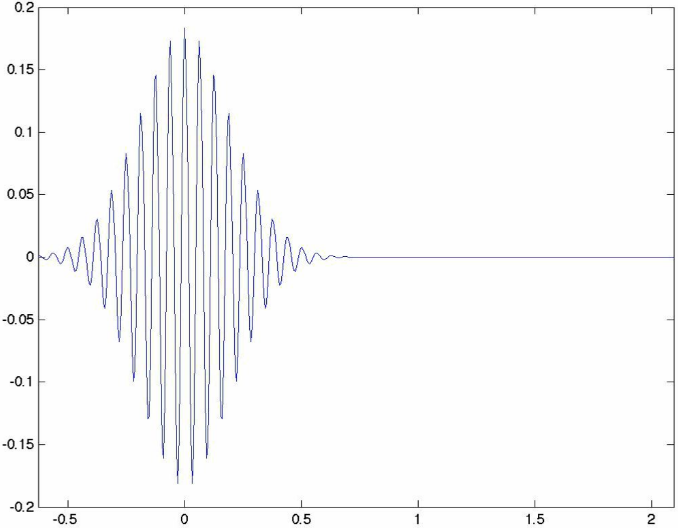
Ondes à Gauche / Droite avec $\Omega(k) = \sqrt{g\,k\,\tanh(k\,h_r)}$

$\widehat{\eta}_0(k_x) = \widehat{\eta}_{G0}(k_x) + \widehat{\eta}_{D0}(k_x)$ et $\widehat{\Phi}_0(k_x) = \widehat{\Phi}_{G0}(k_x) + \widehat{\Phi}_{D0}(k_x)$

Cas d'un paquet d'ondes à droites ($\widehat{\eta}_{G0} = 0$)

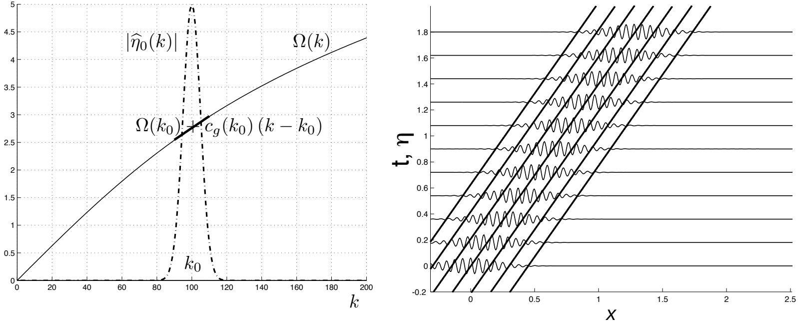
$$\eta(x,t) = \int_{\mathbf{R}^+} \widehat{\eta}_0(k_x)\, e^{i\,k_x\,x - i\,\Omega(k)\,t}\; dk_x + c.c.$$

$$\phi(x,z,t) = \int_{\mathbf{R}^+} \frac{g\,\widehat{\eta}_0(k_x)}{i\,\Omega(k)}\, \frac{\cosh[k\,(z+h_r)]}{\cosh(k\,h_r)}\, e^{i\,k_x\,x - i\,\Omega(k)\,t}\; dk_x + c.c.$$


0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.5
0
0.5
1
1.5
2

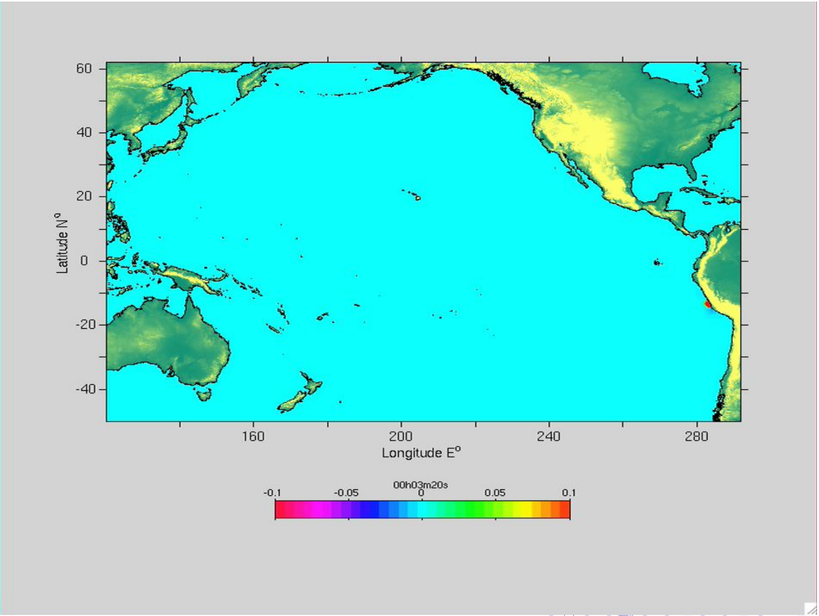
Condition initiale $\eta_0(x) = 2\,\eta_p\ \cos(k_0\,x)\ E(x)$ avec

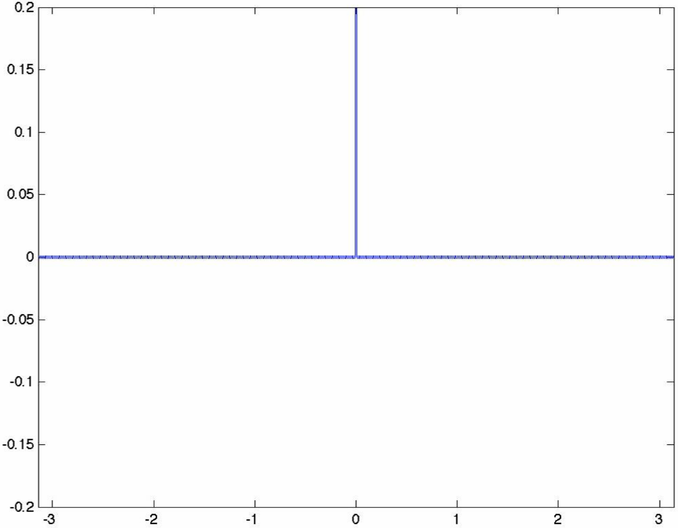
$$\widehat{E}(q) = \exp\left(\frac{-q^2}{2\chi^2}\right) \iff E(x) = \chi\,\sqrt{2\,\pi}\ \exp\left(-\frac{\chi^2\,x^2}{2}\right)$$



Paquet d'ondes : $\Omega(k) = \Omega(k_0) + c_g(k_0)(k - k_0) + \dots$

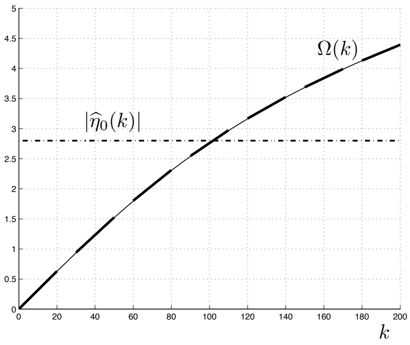
$$\eta(x,t) = 2\,\eta_p\ \cos[k_0\,x - \Omega(k_0)\,t]\ E[x - c_g(k_0)\,t]$$

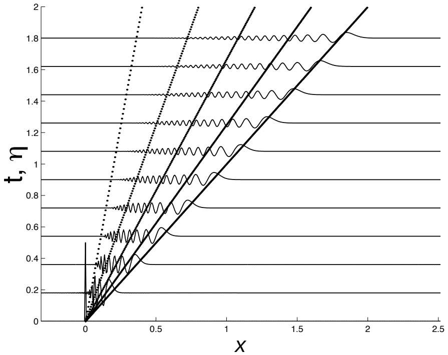

60
40
20
0
-20
-40
Latitude N°
160
200
240
280
Longitude E°
00h03m20s
-0.1
-0.05
0
0.05
0.1


0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-3
-2
-1
0
1
2
3

Condition initiale

$$\widehat{\eta}_0(k) = \eta_m \quad \Longleftrightarrow \quad \eta_0(x) = 2\,\pi\,\eta_m\,\delta(x)$$





Réponse impulsionnelle $\Omega(k) = \sqrt{g\,k\,\tanh(k\,h_r)}$

$$\eta(x,t) = \eta_m \int_{\mathbf{R}^+} e^{i\,k_x\,x - i\,\Omega(k)\,t}\,dk_x + c.c. \quad \text{avec} \quad k = |k_x|$$

Comportement le long de la trajectoire $x = c\,t$

$$I(t) = \eta(c\,t, t) = \eta_m \int_{\boldsymbol{R}^+} e^{i\,[k_x\,c - \Omega(k)]\,t}\,dk_x + c.c.$$

Méthode de la phase stationnaire : $I(t) = \int g(k)\,\exp[i\,\psi(k)\,t]\,dk$

$$\begin{array}{rcl} \text{si } \Psi \text{ est monotone} & : & I(t) \quad \text{décroît exponentiellement} \\ \text{si } k_* \text{ tel que } \Psi'(k_*) = 0 & : & I(t) \sim \dfrac{1}{\sqrt{t}}\,G(k_*)\,e^{i\,\Psi(k_*)\,t} \end{array}$$

Paquet d'onde dispersé : $\Psi(k_x) = k_x\,c - \Omega(k)$

$$\eta(x,t) \sim \frac{1}{\sqrt{t}}\,G\left[k_c\left(\tfrac{x}{t}\right)\right]\;e^{i\,k_c\left(\frac{x}{t}\right)x - \Omega\left[k_c\left(\frac{x}{t}\right)\right]t}$$

où $\quad k_* = k_c(c) \quad$ est le nombre d'onde défini par $\quad c_g(k_*) = c$

et $\quad G(k_*) = \eta_m \sqrt{\frac{2\pi}{|\Omega''(k_*)|}}\;\exp\left\{-i\,\text{sign}[\Omega''(k_*)]\,\frac{\pi}{4}\right\}$