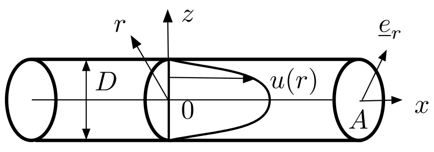
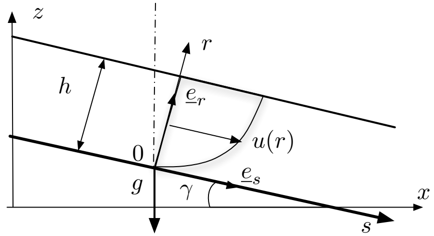
Écoulements incompressibles

MEC567 : SCIENCES DE L'EAU ET ENVIRONNEMENT

Olivier THUAL, 15 janvier 2011





Introduction

1. Cinématique

Un écoulement décrit des particules fluides de vitesse $\underline{U}$ et induit les représentations eulérienne ou lagrangienne des champs. La dérivée particulaire est obtenue en suivant les trajectoires.

2. Lois de conservation

Les théorèmes de transport sur des domaines de particules permettent de définir le tenseur des contraintes et de dériver les lois de conservation de la masse et de la quantité de mouvement.

3. Fluides newtoniens

Le tenseur des contraintes visqueuses est proportionnel au tenseur des taux de déformations pour cette loi rhéologique. Les équations de Navier-Stokes sont illustrées sur deux exemples.

Équations de Navier-Stokes incompressibles

$$\mathrm{div}\ \underline{U} = 0 \qquad \text{et} \qquad \frac{d\underline{U}}{dt} = -\frac{1}{\rho}\,\underline{\mathrm{grad}}\ p + \underline{F} + \nu\,\Delta\underline{U}$$

$\underline{U}(\underline{x},t) = (u,v,w)$ est la vitesse, t le temps, $\underline{x} = (x,y,z)$ l'espace,
ρ est la masse volumique constante, p est la pression,
$\underline{F}$ sont les forces massiques, ν est la viscosité cinématique.

$$\text{Dérivée particulaire : } \frac{d}{dt} = \frac{\partial}{\partial t} + \underline{U}\cdot\underline{\mathrm{grad}}$$

$$\mathrm{div}\ \underline{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad \text{et} \qquad \underline{\mathrm{grad}}\ p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$

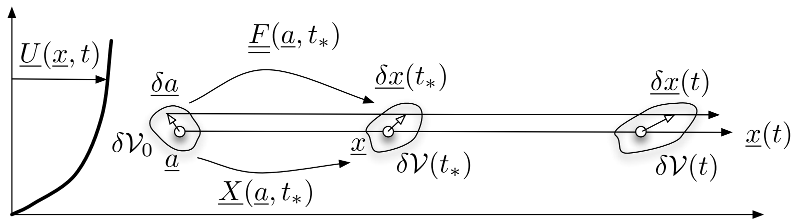
$$\Delta\underline{U} = (\Delta u, \Delta v, \Delta w) \qquad \text{et} \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Ces équations viennent de

- La conservation de la masse et de la quantité de mouvement
- La contrainte d'incompressibilité
- La loi rhéologique des fluides newtoniens

Trajectoires :

$$\frac{d}{dt}\underline{x}(t) = \underline{U}[\underline{x}(t), t] \quad \text{avec} \quad \underline{x}(0) = \underline{a} \qquad \Longleftrightarrow \qquad \underline{x}(t) = \underline{X}(\underline{a}, t)$$



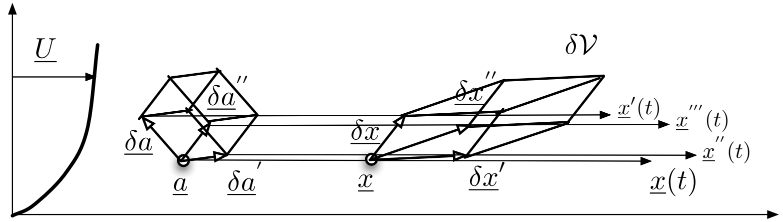
Matrice jacobienne $\underline{\underline{F}}(\underline{a},t)$ avec $F_{ij}(\underline{a},t) = \frac{\partial X_i}{\partial a_j}(\underline{a},t)$

$$\underline{X}(\underline{a} + \underline{\delta a}, t) = \underline{X}(\underline{a},t) + \underline{\underline{F}}(\underline{a},t)\cdot\underline{\delta a} + O(\|\underline{\delta a}\|^2)$$

$$\Longrightarrow \qquad \underline{\delta x}(t) \sim \underline{\underline{F}}(\underline{a},t)\cdot\underline{\delta a}$$

Pour démontrer $\delta\mathcal{V}(t) = \delta\mathcal{V}_0\, J(\underline{a},t)$ avec $J = \det \underline{\underline{F}}$, considérons

$$\delta\mathcal{V}(t) = \left(\underline{\delta x}(t), \underline{\delta x}'(t), \underline{\delta x}''(t)\right) = \begin{vmatrix} \delta x_1 & \delta x_1' & \delta x_1'' \\ \delta x_2 & \delta x_2' & \delta x_2'' \\ \delta x_3 & \delta x_3' & \delta x_3'' \end{vmatrix}$$



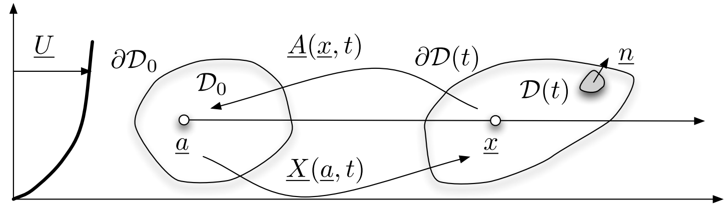
Choisissons $\underline{\delta x}(0) = \delta a\, \underline{e}_x$, $\underline{\delta x}'(0) = \delta a\, \underline{e}_y$ et $\underline{\delta x}''(0) = \delta a\, \underline{e}_z$:

$$\delta\mathcal{V}(t) = \left(\underline{\underline{F}}\cdot\underline{\delta x}(0),\ \underline{\underline{F}}\cdot\underline{\delta x}'(0),\ \underline{\underline{F}}\cdot\underline{\delta x}''(0)\right) = \delta\mathcal{V}(0)\, \det \underline{\underline{F}}$$

Changement de variable $\underline{x} = \underline{X}(\underline{a},t)$ dans un domaine mobile :

$$\iiint_{\mathcal{D}(t)} B(\underline{x},t)\, d^3x = \iiint_{\mathcal{D}_0} B^{(L)}(\underline{a},t)\, J(\underline{a},t)\, d^3a$$

avec $\quad \mathcal{D}(t) = \underline{X}(\mathcal{D}_0,t) \quad$ et $\quad J(\underline{a},t) = \det \underline{\underline{F}}(\underline{a},t)$.

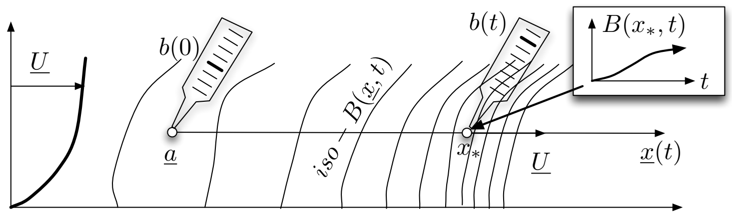


Représentations eulérienne et lagrangienne d'un champ B :

$$B[\underline{X}(\underline{a},t),t] = B^{(L)}(\underline{a},t) \iff B(\underline{x},t) = B^{(L)}[\underline{A}(\underline{x},t),t]$$

Dérivée particulaire :

$$\frac{dB}{dt}(\underline{x},t) = \frac{\partial B}{\partial t}(\underline{x},t) + \underline{U}(\underline{x},t)\cdot \underline{\mathrm{grad}}\, B(\underline{x},t)$$



Champ B mesuré le long d'une trajectoire $\underline{x}(t)$:

$$b(t) = B^{(L)}(\underline{a},t) = B[\underline{x}(t),t]$$

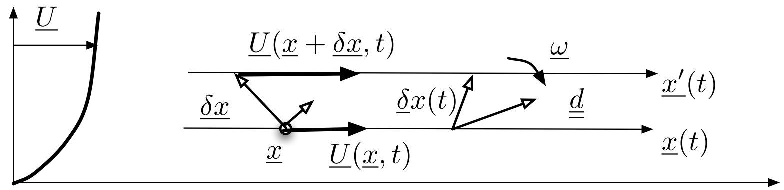
$$\Longrightarrow \quad \frac{db}{dt}(t) = \frac{\partial B^{(L)}}{\partial t}(\underline{a},t) = \left(\frac{\partial B}{\partial t} + \underline{U}\cdot\underline{\mathrm{grad}}\, B\right)[\underline{x}(t),t]$$

Matrice jacobienne du champ de vitesse :

$$\underline{U}(\underline{x}+\underline{\delta x},t) = \underline{U}(\underline{x},t) + \underline{\underline{K}}(\underline{x},t)\cdot\underline{\delta x} + \underline{O}(\|\underline{\delta x}\|^2)$$

$$\underline{\underline{K}} = \underline{\underline{\omega}} + \underline{\underline{d}} \quad \text{avec} \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \quad \text{et} \quad d_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

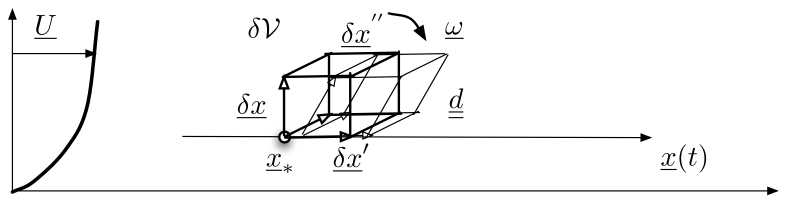
$$\underline{U}(\underline{x}',t) = \underline{U}(\underline{x},t) + \underline{\omega}(\underline{x},t)\wedge(\underline{x}'-\underline{x}) + \underline{\underline{d}}\cdot\underline{\delta x} + \underline{O}(\|\underline{\delta x}\|^2)$$



$$\frac{d}{dt}[\underline{\delta x}(t)] = \underline{U}[\underline{x}'(t),t] - \underline{U}[\underline{x}(t),t] = \underline{\underline{K}}[\underline{x}(t),t]\cdot\underline{\delta x}(t) + \underline{O}[\|\underline{\delta x}(t)\|^2]$$

Expression du taux de variation des volumes $\left(\frac{d}{dt}\delta\mathcal{V}\right)/\delta\mathcal{V} = \text{div }\underline{U}$:

choisissons $\underline{\delta x}(t_*) = \delta x\ \underline{e}_x$, $\quad \underline{\delta x}'(t_*) = \delta x\ \underline{e}_y$ et $\underline{\delta x}''(t_*) = \delta x\ \underline{e}_z$



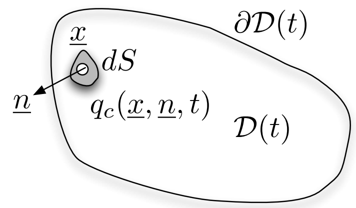
$$\frac{d}{dt}\delta\mathcal{V}(t_*) = \left(\underline{\underline{K}}\cdot\underline{\delta x}, \underline{\delta x}', \underline{\delta x}''\right) + \left(\underline{\delta x}, \underline{\underline{K}}\cdot\underline{\delta x}', \underline{\delta x}''\right) + \left(\underline{\delta x}, \underline{\delta x}', \underline{\underline{K}}\cdot\underline{\delta x}''\right)$$

$$= (\delta x)^3 \left[\begin{vmatrix} K_{11} & 0 & 0 \\ K_{21} & 1 & 0 \\ K_{31} & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & K_{12} & 0 \\ 0 & K_{22} & 0 \\ 0 & K_{32} & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & K_{13} \\ 0 & 1 & K_{23} \\ 0 & 0 & K_{33} \end{vmatrix} \right]$$

$$= (\delta x)^3 \left(K_{11} + K_{22} + K_{33}\right) = \delta\mathcal{V}(t_*)\ \text{div }\underline{U}[\underline{x}(t_*), t_*]$$

Forme générale d'une équation de bilan :

$$\frac{d}{dt}\iiint_{\mathcal{D}(t)} c(\underline{x}, t)\ d^3x + \iint_{\partial\mathcal{D}(t)} q_c(\underline{x}, \underline{n}, t)\ dS = \iiint_{\mathcal{D}(t)} f_c(\underline{x}, t)\ d^3x$$



Si $\mathcal{D}(t)$ est un domaine transporté par le mouvement $\underline{U}$:

$$\frac{d}{dt}\iiint_{\mathcal{D}(t)} c\ d^3x = \iiint_{\mathcal{D}(t)} \frac{\partial c}{\partial t}\ d^3x + \iint_{\partial\mathcal{D}(t)} c\ \underline{U}\cdot\underline{n}\ dS$$

Démonstration :

$$\frac{d}{dt}\iiint_{\mathcal{D}(t)} c(\underline{x}, t)\ d^3x = \frac{d}{dt}\iiint_{\mathcal{D}_0} c^{(L)}(\underline{a}, t)\ J(\underline{a}, t)\ d^3a$$

$$= \iiint_{\mathcal{D}_0}\left(\frac{\partial c^{(L)}}{\partial t}J + c^{(L)}\frac{\partial J}{\partial t}\right) d^3a = \iiint_{\mathcal{D}_0}\left[\left(\frac{dc}{dt}\right)^{(L)} + c^{(L)}(\text{div }\underline{U})^{(L)}\right] J(\underline{a}, t) d^3a$$

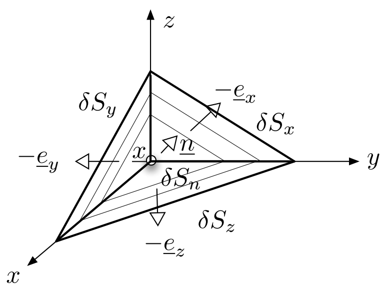
combiner $\delta\mathcal{V}(t) = J(\underline{a}, t)\,\delta\mathcal{V}(0)$ et $\frac{d}{dt}\delta\mathcal{V}(t) = \text{div }\underline{U}[\underline{x}(t), t]\,\delta\mathcal{V}(t)$
pour obtenir $\frac{\partial J}{\partial t}(\underline{a}, t) = \text{div }\underline{U}[\underline{X}(\underline{a}, t), t]\ J(\underline{a}, t)$

$$= \iiint_{\mathcal{D}(t)}\left(\frac{dc}{dt} + c\ \text{div }\underline{U}\right) d^3x = \iiint_{\mathcal{D}(t)}\frac{\partial c}{\partial t}\ d^3x + \iint_{\partial\mathcal{D}(t)} c\ \underline{U}\cdot\underline{n}\ dS$$

utiliser le théorème de la divergence et les relations
$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \underline{U}\cdot\underline{\text{grad}}\ c$ et $\underline{U}\cdot\underline{\text{grad}}\ c + c\ \text{div }\underline{U} = \text{div }(c\ \underline{U})$

Théorème du flux : si pour tout $\mathcal{D}(t)$ on a

$$\frac{d}{dt}\iiint_{\mathcal{D}(t)} c(\underline{x}, t)\ d^3x + \iint_{\partial\mathcal{D}(t)} q_c(\underline{x}, \underline{n}, t)\ dS = \iiint_{\mathcal{D}(t)} f_c(\underline{x}, t)\ d^3x$$



alors : linéarité et vecteur flux

$$q_c(\underline{x}, \underline{n}, t) = \underline{Q}_c(\underline{x}, t)\cdot\underline{n}$$

Démonstration des petits tétraèdres

$$\frac{d}{dt}\iiint_{\mathcal{D}(t)} c(\underline{x}, t)\ d^3x + \iint_{\partial\mathcal{D}(t)} \underline{Q}_c(\underline{x}, t)\cdot\underline{n}\ dS = \iiint_{\mathcal{D}(t)} f_c(\underline{x}, t)\ d^3x$$

Bilan local et loi de conservation

$$\iiint_{\mathcal{D}(t)} \frac{\partial c}{\partial t}\, d^3x + \iint_{\partial\mathcal{D}(t)} (c\,\underline{U} + \underline{Q}_c)\cdot \underline{n}\, dS = \iiint_{\mathcal{D}(t)} f_c\, d^3x$$

$$\frac{\partial c}{\partial t} + \text{div}\,(c\,\underline{U} + \underline{Q}_c) = \frac{dc}{dt} + c\,\text{div}\,\underline{U} + \text{div}\,\underline{Q}_c = f_c$$

Conservation de la masse

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho\,\underline{U}) = \frac{\partial \rho}{\partial t} + \underline{U}\cdot\underline{\text{grad}}\,\rho + \rho\,\text{div}\,\underline{U} = \frac{d\rho}{dt} + \rho\,\text{div}\,\underline{U} = 0$$

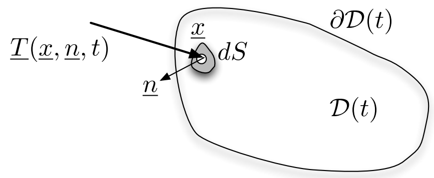
Contrainte isochore $\left[\frac{d}{dt}\delta\mathcal{V}(t)\right]/\delta\mathcal{V}(t) = \text{div}\,\underline{U} = 0$

L'équation de conservation de la masse s'écrit : $\frac{d\rho}{dt} = 0$

Si $\rho(\underline{x},0) = \rho_0$ (homogène) à $t=0$ on a donc $\rho = \rho_0$

Principe fondamental de la dynamique :

$$\frac{d}{dt}\iiint_{\mathcal{D}(t)} \rho\,\underline{U}\,d^3x - \iint_{\partial\mathcal{D}(t)} \underline{T}(\underline{x},\underline{n},t)\,dS = \iiint_{\mathcal{D}} \rho\,\underline{F}\,d^3x$$



Tenseur des contraintes :

$$\underline{T}(\underline{x},\underline{n},t) = \underline{\underline{\sigma}}(\underline{x},t)\cdot\underline{n}$$

$\underline{T}$: forces surfaciques de contact

$\underline{\underline{\sigma}}$: tenseur symétrique

Résultante volumétrique des forces de contact $\underline{\text{div}}\,\underline{\underline{\sigma}}$:

$$\rho\,\frac{d\underline{U}}{dt} = \rho\left(\frac{\partial\underline{U}}{\partial t} + \underline{U}\cdot\underline{\underline{\text{grad}}}\,\underline{U}\right) = \rho\,\underline{F} + \underline{\text{div}}\,\underline{\underline{\sigma}}$$

$i^{\text{ème}}$ composante du vecteur $\underline{\text{div}}\,\underline{\underline{\sigma}}$: $\frac{\partial\sigma_{ij}}{\partial x_j} =: \sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j}$

Loi rhéologique des fluides newtoniens

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \underline{\underline{\tau}} = -p\,\underline{\underline{I}} - \frac{2\,\mu}{3}\,\text{div}\,\underline{U}\,\underline{\underline{I}} + 2\,\mu\,\underline{\underline{d}}$$

Pour les fluides incompressibles, la contrainte div $\underline{U} = 0$ conduit à :

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 2\,\mu\,\underline{\underline{d}} \qquad \Longrightarrow \qquad \underline{\text{div}}\,\underline{\underline{\sigma}} = -\underline{\text{grad}}\,p + \mu\,\Delta\underline{U}$$

Équations de Navier-Stokes incompressibles ($\nu = \mu/\rho$)

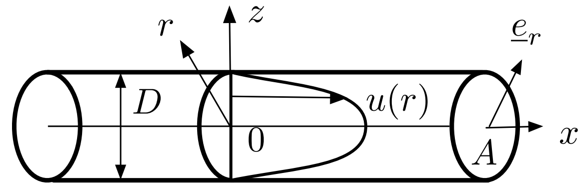
$$\text{div}\,\underline{U} = 0 \qquad \text{et} \qquad \frac{d\underline{U}}{dt} = -\frac{1}{\rho}\,\underline{\text{grad}}\,p + \underline{F} + \nu\,\Delta\underline{U}$$

Conditions aux limites :

- Rigides : $\underline{U} = \underline{0}$
- Libres : $\underline{U}\cdot\underline{n} = 0$ et $\underline{\underline{\sigma}}\cdot\underline{n} - (\underline{n}\cdot\underline{\underline{\sigma}}\cdot\underline{n})\;\underline{n} = \underline{0}$
- Surface libre : $\frac{dF}{dt} = 0$ et $\underline{\underline{\sigma}}\cdot\underline{n} = -p_a\,\underline{n}$ en $F(\underline{x},t) = 0$

Solution laminaire $\underline{U} = u(r)\,\underline{e}_x$

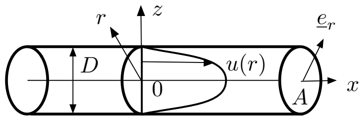
Conditions aux limites rigides : $u = 0$ en $r = D/2$



$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u\,, \qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{et} \qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Forçage par un gradient de pression constant G :

$$p(\underline{x}) = p_r - G\,x - \rho\,g\,z\,, \qquad u(r) = \frac{G}{\rho\,\nu}\left(\frac{D^2}{16} - \frac{r^2}{4}\right)$$



Vitesse moyenne : $U = \dfrac{1}{A}\displaystyle\int_{\mathcal{A}} u \; da$

Charge hydraulique H et perte de charge linéique J

$$H = \frac{p}{\rho\, g} + z + \frac{u^2}{2\, g} \,, \qquad J = \frac{1}{A}\int_{\mathcal{A}} \frac{1}{g}\left(-\nu\, \Delta u\right) \; da$$

$$\frac{\partial H}{\partial x} = -J \qquad \Longrightarrow \qquad \frac{G}{\rho\, g} = J \,.$$

$$J = \lambda\, \frac{U^2}{2\, g\, D} \quad \text{avec} \quad \lambda = \frac{64}{Re} \quad \text{et} \quad Re = \frac{U\, D}{\nu}$$

Conductivité hydraulique K_p

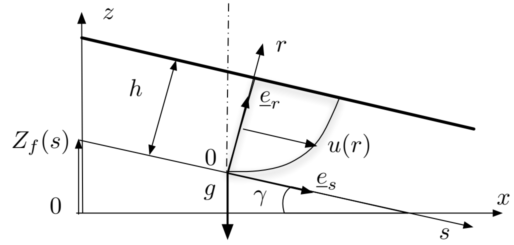
$$U = -K_p\, \frac{dH}{dx} \qquad \text{avec} \qquad K_p = \frac{g D^2}{32\, \nu}$$

Solution laminaire $\underline{U} = u(r)\, \underline{e}_s$. Conditions aux limites :

$$\left(u = 0 \quad \text{en} \quad r = 0\right) \qquad \text{et} \qquad \left(\underline{\underline{\sigma}} \cdot \underline{e}_r = -p_a\, \underline{e}_r \quad \text{en} \quad r = h\right)$$

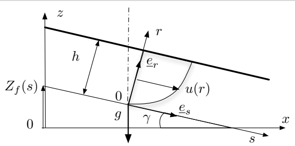
Tenseur des contraintes :

$$\begin{aligned}
\underline{\underline{\sigma}} &= \quad -p\, \underline{\underline{I}} + 2\, \rho\, \nu\, \underline{\underline{d}} \\
\underline{\underline{d}} &= \tfrac{1}{2}\, \tfrac{du}{dr}\left(\underline{e}_r \otimes \underline{e}_s + \underline{e}_s \otimes \underline{e}_r\right) \\
&= \quad \tfrac{1}{2}\, \tfrac{du}{dr} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{(\underline{e}_s, \underline{e}_r)}
\end{aligned}$$



$$0 = g\, I + \nu\, \frac{d^2 u}{dr^2}(r) \quad \text{et} \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}(x,z) - g \quad \text{avec} \quad I = \sin\gamma$$

$$\frac{p}{\rho\, g} + z = \frac{p_a}{\rho\, g} + h\, \cos\gamma + Z_f(s) \qquad \text{et} \qquad u = \frac{gI}{\nu}\left(h\, r - r^2/2\right)$$



Vitesse moyenne : $U = \dfrac{1}{h}\displaystyle\int_0^h u \; dr$

Charge hydraulique H et perte de charge linéique J

$$H = \frac{p_a}{\rho\, g} + h\, \cos\gamma + Z_f(s) + \frac{u^2}{2\, g} \,, \qquad J = \frac{1}{h}\int_0^h \frac{1}{g}\left(-\nu\, \frac{d^2 u}{dr^2}\right) \; dr$$

$$\frac{\partial H}{\partial s} = -J \qquad \Longrightarrow \qquad I = J$$

$$J = \lambda\, \frac{U^2}{2gD_H} \,, \qquad \lambda = \frac{96}{Re} \,, \qquad Re = \frac{U\, D_H}{\nu} \qquad \text{et} \qquad D_H = 4\, h$$

Contrainte de cisaillement $\tau_* = \underline{e}_s \cdot \underline{\underline{\sigma}} \cdot \underline{e}_r$ en $z = Z_f(s)$

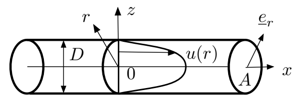
$$\tau_* = \rho\, g\, R_H\, J \qquad \text{avec} \qquad R_H = h$$

Charge hydraulique H et perte de charge linéique J

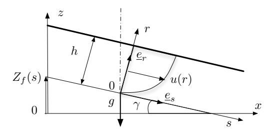
$$H = \frac{p}{\rho\, g} + z + \frac{u^2}{2\, g} \,, \qquad J = \frac{1}{A}\int_{\mathcal{A}} \frac{1}{g}\left(-\nu\, \Delta u\right) \; dA$$

Relation de Darcy-Weissbach et contrainte de cisaillement

$$J = \lambda\, \frac{U^2}{2\, g\, D_H} \,, \qquad \tau_* = \rho\, g\, R_H\, J \qquad \text{avec} \qquad D_H = 4\, R_H$$



$$Re = \frac{U\, D_H}{\nu}$$



$$\lambda = \frac{64}{Re} \,, \qquad D_H = D$$

$$\lambda = \frac{96}{Re} \,, \qquad D_H = h$$