COURSE 3

GENERIC INSTABILITIES AND NONLINEAR DYNAMICS

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1. Introduction

Quite a wide range of concepts and techniques on nonlinear dynamics has been presented by Ed Spiegel in this school. We note in particular that the study of multiple instabilities gives useful insight in the nature of chaotic dynamics of a physical system, even far from the high-codimension surface in the parameter space. We want to consider now a different point of view: what are the most current instabilities one can encounter when varying a parameter in a physical experiment? We will forget here the high-codimension situations, and concentrate on the "generic" instabilities of stationary solutions, given a "generic" path in the parameter space.

Genericity is a property of a phenomenon, here instability, that must always refer to a given class of problems. For instance, a phenomenon can be nongeneric in the wide class of all the dynamical systems, but generic in the class of all the Hamiltonian systems. The aim of this course is to describe the most common generic destabilizations of an equilibrium that arise in physical systems. Here, the emphasis will be on systems governed by evolution equations of the type of those encountered in hydrodynamics. Such equations are invariant under usual symmetries: space or time translations and reflection symmetries. We will also insist on those cases where the equilibrium is itself invariant under some of these symmetries.

The material in this course is voluntarily simple or schematical. More advanced developments may be found in refs. [1–3] for section 2, and refs. [6–10] for sections 3 and 4.

In section 2 we recall some elementary results on the destabilization of a fixed point in a dissipative dynamical system. In this case the generic bifurcations are the saddle node and the Hopf bifurcations, but we also study the pitchfork bifurcation, generic in the class of systems with a reflection symmetry. We give the normal forms of the pitchfork and the Hopf bifurcations, and present a method to calculate explicitly the nonlinear coefficients. This method is called the Krylov–Bogolyubov–Mitropolsky (KBM) method by the authors of ref. [3], who extended it to competing instabilities. This section is illustrated by two examples, on which we apply the different concepts: the Lorenz model [4] and the Liénard oscillators [5].

In section 3 we study the destabilization of the equilibrium with partial differential equations involving time and space. Most of the concepts are presented with examples: 2D Rayleigh–Bénard convection, spatially coupled oscillators, and thermohaline convection. An important factor in a physical system is the existence of symmetries (space or time translation, space reflection). The destabilizations are strongly influenced by these symmetries (see refs. [6-8]). When the problem has a confined geometry, there is a discrete set of modes. We are then led to consider the destabilization of a system with ordinary differential equations (dynamical system), obtained by truncation of the mode decompositions. The generic instabilities of such systems differ from the generic instabilities of a randomly chosen dynamical system: additional symmetries are due to the symmetries of the original physical system and to the decomposition of real fields in complex modes. We try a classification of the "most generic" destabilizations, with a simple model. We investigate the normal forms (amplitude equations) of these instabilities. When the box is large, these amplitude equations are partial differential equations.

In section 4 we give an introduction to the phase equation theory (see ref. [9]). We first examine the existence of families of periodic structures in physics, and part them into two classes, as in ref. [10]: the dispersive ones (A type) and the nondispersive ones (B type). These periodic structures are likely to be destabilized by phase modes, associated with the symmetries of the equations they are solutions of. We show how to derive phase equations from the Landau or Landau–Ginzburg equation to describe the phase instabilities of the periodic solutions of these amplitude equations (see also refs. [6–8]). We show how to derive such phase equations by symmetry considerations, given a general problem admitting periodic structures and symmetries.

2. Bifurcations in dissipative dynamical systems

2.1. Examples of dissipative dynamical systems

In this section we present elementary concepts concerning bifurcations of a fixed point in dissipative dynamical systems, normal forms, and practical methods for their calculation.

2.1.1. Definition

We consider a general system of ordinary differential equations with N degrees of freedom:

$$\dot{X} = F_\lambda(X), \qquad \text{with} \quad X(t) \in \boldsymbol{R}^N. \tag{2.1}$$

The word dynamical system is generally reserved for sufficiently small values of N. In mechanical problems one often encounters conservative dynamical systems,

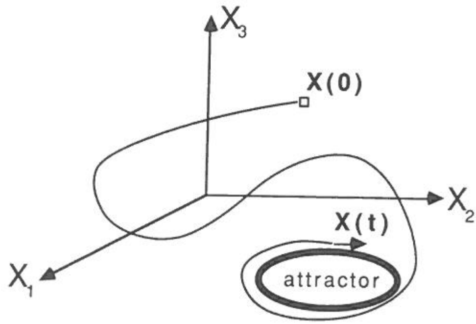

Fig. 1. A trajectory in phase space.

admitting Lagrangian or Hamiltonian functions. We will consider here only dissipative dynamical systems: the volumes in phase space are contracting. We then look at trajectories and their attractors in phase space (fig. 1). When the control parameters, λ, vary, destabilizations of these attractors may occur with topological changes in the phase portraits: these are called bifurcations.

2.1.2. Example: the Lorenz model

As a typical example of dissipative dynamical systems, we first consider the model which was studied by Lorenz:

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{2.2}$$

with $\sigma > 0$, $b > 0$, and $r > 0$. This dynamical system may be derived from the Boussinesq equations, which govern the convective motions in a fluid heated from below. σ is the Prandtl number and b is a geometrical factor of the box. We take as control parameter r, the ratio of the Rayleigh number to the critical Rayleigh number.

2.1.3. Example: the Liénard oscillators

A very general equation for oscillators has been investigated by Liénard:

$$\ddot{u} + R(u)\dot{u} + P(u) = 0, \tag{2.3}$$

where $R(u)$ is an even function of u while $P(u)$ is odd. When the oscillations are small we replace these functions by the first two orders of their Taylor series:

$$\begin{aligned} R(u) &= -\epsilon + ru^2, \\ P(u) &= \omega^2 u + pu^3. \end{aligned} \tag{2.4}$$

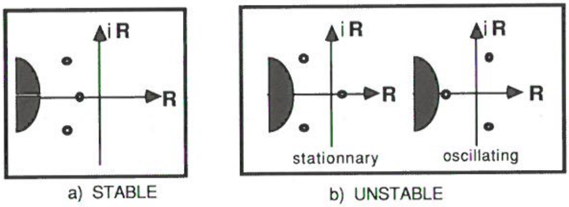

Fig. 2. Spectrum of L in $\boldsymbol{C}$.

Here we consider ϵ as the control parameter, and study the destabilization of $u = 0$. For $R = 0$ one recognizes the Duffing oscillator, while the case $p = 0$ stands for the van der Pol oscillator (the standard form $r = \epsilon$ is recovered by a scaling of u). The small-oscillations Linéard equations are also the normal-form equations of a codimension-two instability whose singular operator is a Jordan bloc [3].

It is convenient to write this Liénard equation in the standard form for dynamical systems:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 x + \epsilon y - r x^2 y - p x^3. \end{aligned} \tag{2.5}$$

2.2. *Bifurcations of a fixed point*

2.2.1. *Stability of a fixed point*

A fixed point (equilibrium) of a dynamical system is a point X_λ in phase space such that $F_\lambda(X_\lambda) = 0$ in eq. (2.1).

To study its stability, we need to investigate the eigenvalues of the $N \times N$ operator obtained by linearizing the equations around X_λ:

$$\begin{aligned} X &= X_\lambda + U, \\ \dot{U} &= L_\lambda U + N_\lambda(U), \qquad \text{with} \quad L_\lambda = \mathrm{D}F_\lambda(X_\lambda). \end{aligned} \tag{2.6}$$

The fixed point is stable when all the eigenvalues of the operator L_λ (the differential of F_λ at X_λ) have a negative real part. An instability occurs when one real eigenvalue or two complex conjugated eigenvalues, for the generic case (more for nongeneric cases), cross the imaginary axis (fig. 2). The topological singularity associated with an instability is called bifurcation.

2.2.2. *Real-eigenvalue crossing*

The generic bifurcation when a real value crosses the imaginary axis, is the saddle-node bifurcation: two fixed points, one unstable and the studied stable one, collapse at the critical value of the parameter. There is a change in the number of

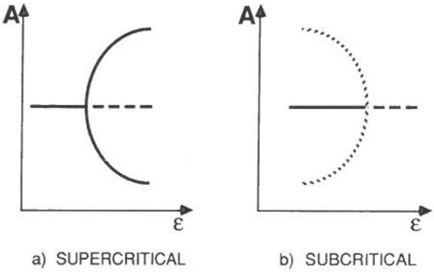

Fig. 3. Bifurcation diagram of the pitchfork bifurcation: (a) supercritical case, $\alpha > 0$, (b) subcritical case, $\alpha < 0$.

solutions of the equation $F_\lambda(X) = 0$, because the X-differential of F is noninvertible: this violates the implicit-functions theorem.

When there is a symmetry in the equations (nongeneric case), there may be a pitchfork bifurcation present. A simple example can be seen with the following, simple dynamical system:

$$\dot{A} = \epsilon A - \alpha A^3, \tag{2.7}$$

with A and α real. Here ϵ is the control parameter, which is critical at 0. Depending on the sign of α, the pitchfork bifurcation is supercritical or subcritical (fig. 3).

2.2.3. Complex conjugated eigenvalues

The generic situation is a Hopf bifurcation. There is no violation of the implicit-function theorem since L_λ remains invertible. But at the bifurcation appears a periodic trajectory centered at the fixed point, with a radius growing from zero. A simple example of a Hopf bifurcation is given by:

$$\dot{W} = (\epsilon + \mathrm{i}\omega)W - \alpha|W|^2 W, \tag{2.8}$$

with W and α complex. Depending on the sign of Re(α), the bifurcation is supercritical or subcritical at $\epsilon = 0$ (fig. 4).

2.2.4. Example: the Lorenz model

The differential of F_λ is an operation which associates to each point $X(x, y, z)$ a 3×3 matrix:

$$\mathrm{D}F_\lambda(X) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}. \tag{2.9}$$

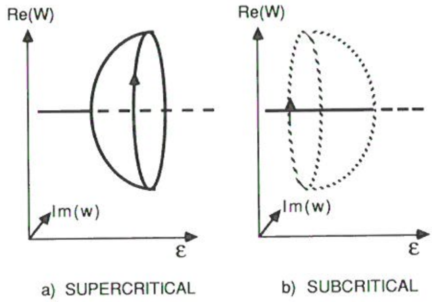

Fig. 4. Bifurcation diagram of the Hopf bifurcation: (a) supercritical case, $\alpha > 0$, (b) subcritical case, $\alpha < 0$.

For $X = 0$ the eigenvalues are found by solving the characteristic equation:

$$(s + b)[s^2 + (\sigma + 1)s + \sigma(1 - r)] = 0. \tag{2.10}$$

When $r = 1$, the null solution gives a pitchfork bifurcation. The equations are invariant under the symmetry $[x \to -x,\ y \to -y,\ z \to z]$. For $r > 1$ the null solution is unstable and the two symmetric solutions $X^+(a,\ a,\ r - 1)$ and $X^-(-a,\ -a,\ r - 1)$, with $a = \sqrt{b(r-1)}$, are stable. These new fixed points correspond physically to convective rolls (rolling clockwise or counterclockwise).

2.2.5. *Example: the Liénard oscillators*

The differential of F_λ applied to the fixed point $X = 0$ reads:

$$\mathrm{D}F_\lambda(X_\lambda) = \begin{pmatrix} 0 & 1 \\ \omega^2 & \epsilon \end{pmatrix}. \tag{2.11}$$

The characteristic polynomial is:

$$s^2 - \epsilon s + \omega^2 = 0. \tag{2.12}$$

When ϵ becomes positive, the stable fixed point $X = 0$ is destabilized by a pair of complex conjugated eigenvalues crossing at $\mathrm{i}\omega$ and $-\mathrm{i}\omega$. It is not trivial to decide whether the Hopf bifurcation is supercritical or subcritical. The calculation of the normal form associated with this bifurcation will give the answer to this question.

2.3. *Normal forms of the bifurcations*

2.3.1. *Pitchfork bifurcation*

We have seen that a pitchfork bifurcation occurs when there is a real eigenvalue crossing the imaginary axis at $\lambda = \lambda_c$, with, in addition, a particular symmetry of the equations. Let:

$$\begin{aligned} &L, \text{ the critical operator: } && L = L_{\lambda_c}, \\ &\phi, \text{ the marginal mode: } && L\phi = 0, \\ &\phi_i, \text{ the damped modes: } && L\phi_i = s_i\phi_i. \end{aligned} \tag{2.13}$$

For any λ we decompose $U(t)$ in the basis of L:

$$U(t) = A(t)\phi + \sum_i B_i(t)\phi_i. \tag{2.14}$$

Substitution in the equation $\dot{U} = L_\lambda U + N_\lambda(U)$ gives:

$$\begin{aligned} \dot{A} &= \mu A + g(A, B), \\ \dot{B}_i &= s_i B_i + g_i(A, B), \end{aligned} \tag{2.15}$$

where μ is the value of the destabilizing eigenvalue and is equivalent to $(\lambda - \lambda_c)$ in the vicinity of λ_c. The characteristic evolution time of the damped modes is of order one, while the evolution time of the marginal mode is of order μ^{-1}. For this reason we suppose that B follows the evolution of A adiabatically, $B = h(A)$, or, more rigorously, invoke the central-manifold theorem:

$$B = h(A) = h_2A^2 + h_3A^3 + O(A^4). \tag{2.16}$$

We then can eliminate B in the evolution equation of A:

$$\dot{A} = \mu A + g\,[A, h(A)] =: \mu A + f(A). \tag{2.17}$$

Because of the symmetry $A \to -A$ leading to a pitchfork bifurcation, the asymptotic expansion of f starts with a cubic term: $f(A) = -\alpha A^3 + O(A^5)$. We have then obtained the normal form of the pitchfork bifurcation at its lowest order:

$$\dot{A} = \mu A - \alpha A^3. \tag{2.18}$$

This is the well-known Landau equation.

2.3.2. The Hopf bifurcation

Two complex conjugated eigenvalues $i\omega$ and $-i\omega$ are crossing the imaginary axis at $\lambda = \lambda_c$. We still use the notations:

$$\begin{aligned} & && L = L_{\lambda_c}, \\ &\phi \text{ and } \overline{\phi}, \text{ the marginal modes: } && L\phi = i\omega, \\ & \qquad\qquad\qquad\qquad\text{and} && L\overline{\phi} = i\omega\overline{\phi}, \\ &\phi_i, \quad \text{the damped modes: } && L\phi_i = s_i\phi_i. \end{aligned} \tag{2.19}$$

For any λ we decompose $U(t)$ in the basis of L:

$$U(t) = W(t)\phi + \overline{W}(t)\overline{\phi} + \sum_i B_i(t)\phi_i. \tag{2.20}$$

We recall that the solutions of the linear problem $\dot{U} = LU$ is a family of ellipses:

$$U(t) = 2a\cos(\omega t - \varphi)V_\mathrm{r} - 2a\sin(\omega t - \varphi)V_\mathrm{i}, \tag{2.21}$$

with $\phi = V_\mathrm{r} + \mathrm{i}V_\mathrm{i}$.

The nonlinear problem $\dot{U} = L_\lambda U + N_\lambda(U)$ leads to:

$$\begin{aligned} \dot{W} &= (\mu + \mathrm{i}\omega)W + g(W, \overline{W}, B), \\ \dot{\overline{W}} &= (\mu - \mathrm{i}\omega)\overline{W} + \overline{g}(W, \overline{W}, B), \\ \dot{B}_i &= s_i B_i + g_i(W, \overline{W}, B), \end{aligned} \tag{2.22}$$

where μ is the real value of the destabilizing eigenvalues and is proportional to $\lambda - \lambda_\mathrm{c}$. The central-manifold expansion reads:

$$\begin{aligned} B = h(W, \overline{W}) &= h_2^{20}\, W^2 + h_2^{11}\, W\overline{W} + h_2^{02}\, \overline{W}^2 + h_3^{30}\, W^3 + h_3^{21}\, W^2\overline{W} \\ &\quad + h_3^{12}\, W\overline{W}^2 + h_3^{21}\, \overline{W}^3 + \mathrm{O}(|W|^4). \end{aligned} \tag{2.23}$$

We then can eliminate B in the evolution equation of W:

$$\dot{W} = (\mu + \mathrm{i}\omega)W + g[W, \overline{W}, h(W, \overline{W})] =: (\mu + \mathrm{i}\omega)W + f(W, \overline{W}). \tag{2.24}$$

The asymptotic expansion of f starts with a quadratic term:

$$\begin{aligned} f(W, \overline{W}) &= f_2^{20}\, W^2 + f_2^{11}\, W\overline{W} + f_2^{02}\, \overline{W}^2 + f_3^{30}\, W^3 + f_3^{21}\, W^2\overline{W} \\ &\quad + f_3^{12}\, W\overline{W}^2 + f_3^{21}\, \overline{W}^3 + \mathrm{O}(|W|^4), \end{aligned} \tag{2.25}$$

but with an arbitrary nonlinear change of variable:

$$W' = W + \psi\left(W, \overline{W}\right), \tag{2.26}$$

with $\psi(W, \overline{W}) = \psi_2^{20}\, W^2 + \psi_2^{11}\, W\overline{W} + \psi_2^{02}\, \overline{W}^2 + \psi_3^{30}\, W^3 + \psi_3^{21}\, W^2\overline{W} + \psi_3^{12}\, W\overline{W}^2 + \psi_3^{21}\, \overline{W}^3 + \mathrm{O}(|W|^4)$, we can eliminate all the coefficients of the asymptotic expansion of f, except the resonant terms. This leads to the normal form of the Hopf bifurcation, which reads in lowest order:

$$\dot{W} = (\mu + \mathrm{i}\omega)W - \alpha|W|^2 W, \tag{2.27}$$

with α complex.

2.4. KBM method for the calculation of normal forms

The calculation of normal forms can be performed either by asymptotic methods involving small parameters (Poincaré–Lindstedt, multiple scale, etc.) or by a direct method called the Krylov–Bogolyubov–Mitropolsky (KBM) method.

2.4.1. Hopf bifurcation

Given a dynamical system $\dot{U} = L_\lambda U + N_\lambda(U)$ encountering a Hopf bifurcation, we aim to calculate the complex coefficient α of the Hopf normal form. Since the KBM method is used at $\lambda = \lambda_c$, we use the notations $L = L_{\lambda_c}$ and $N = N_{\lambda_c}$. The method is contained in the following assumptions:

$$\begin{aligned}
&U(t) = W(t)\phi + \overline{W}(t)\overline{\phi} + V(W, \overline{W}),\\
&V(W, \overline{W}) = V_2^{20}\, W^2 + V_2^{11}\, W\overline{W} + V_2^{02}\, \overline{W}^2 + V_3^{30}\, W^3\\
&\qquad + V_3^{21}\, W^2\overline{W} + V_3^{12}\, W\overline{W}^2 + V_3^{03}\, \overline{W}^3 + \mathrm{O}(|W|^4),\\
&\dot{W} = \mathrm{i}\omega W - \alpha |W|^2 W,\\
&N(U) = N_2^{20}\, W^2 + N_2^{11}\, W\overline{W} + N_2^{02}\, \overline{W}^2 + N_3^{30}\, W^3\\
&\qquad + N_3^{21}\, W^2\overline{W} + N_3^{12}\, W\overline{W}^2 + N_3^{03}\, \overline{W}^3 + \mathrm{O}(|W|^4),\\
&\dot{U} = LU + N(U).
\end{aligned} \tag{2.28}$$

We obtain at each order in $|W|$ a system of recurrent linear equations for the coefficients of V.

– *Order 1*:

$$\begin{aligned}
&- L\phi = \mathrm{i}\omega\phi,\\
&- L\overline{\phi} = -\mathrm{i}\omega\overline{\phi};
\end{aligned} \tag{2.29}$$

this is trivially satisfied.

– *Order 2*:

$$\begin{aligned}
&(2\mathrm{i}\omega - L)V_2^{20} = N_2^{20}(\phi, \overline{\phi}),\\
&- LV_2^{11} = N_2^{11}(\phi, \overline{\phi}),\\
&(-2\mathrm{i}\omega - L)V_2^{02} = N_2^{02}(\phi, \overline{\phi}),
\end{aligned} \tag{2.30}$$

which determines V_2.

– *Order 3*:

$$\begin{aligned}
&(3\mathrm{i}\omega - L)V_3^{30} = N_3^{30}(\phi, \overline{\phi}, V_2),\\
&(\mathrm{i}\omega - L)V_3^{21} = N_3^{21}(\phi, \overline{\phi}, V_2) + \alpha\, \phi,\\
&(-\mathrm{i}\omega - L)V_3^{12} = N_3^{12}(\phi, \overline{\phi}, V_2) + \overline{\alpha}\, \overline{\phi},\\
&(-3\mathrm{i}\omega - L)V_3^{03} = N_3^{03}(\phi, \overline{\phi}, V_2).
\end{aligned} \tag{2.31}$$

The matrix $\mathrm{i}\omega - L$ is not invertible. The calculation of V_3^{21} leads to the compatibility condition:

$$N_3^{21} + \alpha\phi \in \mathrm{Im}(\mathrm{i}\omega I - L). \tag{2.32}$$

This determines a unique value of α. To calculate it, from a practical point of view, we have to define an adjoint problem.

We are here confronted with the Fredholm alternative $MV = Y$, such that the kernel of the linear matrix M is spanned by one vector ϕ. The compatibility condition is easily expressed if we define a duality product or a scalar product. Let ϕ^+ be the spanning vector of the kernel of the transposed operator M^+ (the adjoint operator in the case of a scalar product): $M^+\phi^+ = 0$. The vector Y is in the image $\mathrm{Im}(M)$ if and only if it is orthogonal to ϕ^+:

$$Y \in \mathrm{Im}(M) \Longleftrightarrow \langle Y, \phi^+\rangle = 0. \tag{2.33}$$

In the case of our problem α is then given by:

$$\alpha = -\frac{\langle N_3^{21}, \phi^+\rangle}{\langle \phi, \phi^+\rangle}. \tag{2.34}$$

2.4.2. Pitchfork bifurcation

Given a dynamical system $\dot{U} = L_\lambda U + N_\lambda(U)$ encountering a pitchfork bifurcation, we aim to calculate the real coefficient α in the Landau equation. The KBM method is applied at $\lambda = \lambda_c$, so that we use the same notations L and N as before. We set:

$$\begin{aligned}
&U(t) = A(t)\,\phi + V(A),\\
&V(A) = V_2A^2 + V_3A^3 + \mathrm{O}(A^4),\\
&\dot{A} = -\alpha A^3 + \mathrm{O}(A^5),\\
&N(U) = N_2A^2 + N_3A^3 + \mathrm{O}(A^4),\\
&\dot{U} = LU + N(U).
\end{aligned} \tag{2.35}$$

Again, we obtain at each order in A a system of recurrent linear systems for the coefficients of V.

– *Order 1*: the equation $-L\phi = 0$ is trivially satisfied.

– *Order 2*:

$$-LV_2 = N_2(\phi). \tag{2.36}$$

At first glance, one would expect that there is a compatibility condition to satisfy, since L is noninvertible. If that were true, we should have included a quadratic term in the normal form. But because of the symmetry of the equation, responsible for the occurrence of the pitchfork bifurcation, the vector N_2 is in the image of L. The vector V_2 can be solved.

– *Order 3*:

$$-LV_3 = N_3(\phi) + \alpha\phi. \tag{2.37}$$

The solvability condition of the Fredholm alternative for V_3^{21} leads to:

$$N_3 + \alpha\phi \in \mathrm{Im}(L). \tag{2.38}$$

This gives a unique value for α. With the definition of a scalar (or duality) product we can express α:

$$\alpha = -\frac{\langle N_3, \phi^+\rangle}{\langle \phi, \phi^+\rangle}. \tag{2.39}$$

2.4.3. Application to the Lorenz model

The critical control parameter is $r = 1$:

$$L = \mathrm{D}F_{\lambda_c}(0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}. \tag{2.40}$$

$$L\phi = 0, \qquad \text{with} \quad \phi = (1, 1, 0). \tag{2.41}$$

Since $N(U)$ is quadratic, we can define a quadratic form Q:

$$Q(U_1, U_2) = [0, -\tfrac{1}{2}(x_1 z_2 + x_2 z_1), \tfrac{1}{2}(x_1 y_2 + x_2 y_1)], \tag{2.42}$$

with $N(U) = Q(U, U)$.

We will need the explicit values of N_2 and N_3:

$$\begin{aligned} U(t) &= A(t)\phi + V_2 A^2 + V_3 A^3 + \mathrm{O}(A^4), \\ N(U) &= A^2 Q(\phi, \phi) + 2A^3 Q(\phi, V_2) + \mathrm{O}(A^4), \\ Q(\phi, \phi) &= (0, 0, 1). \end{aligned} \tag{2.43}$$

We choose the canonical scalar product in $\boldsymbol{R}^3$ to solve the Fredholm alternatives:

$$L^+ = \begin{pmatrix} -\sigma & -1 & 0 \\ \sigma & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}. \tag{2.44}$$

$$L^+\phi^+ = 0, \quad \text{with} \qquad \phi^+ = (1, \sigma, 0). \tag{2.45}$$

We then apply the KBM algorithm:

– *Order 2*:

$$-LV_2 = Q(\phi, \phi) = (0, 0, 1). \tag{2.46}$$

We check that $Q(\phi, \phi)$ is indeed in the kernel of L^+:

$$\langle Q(\phi, \phi), \phi^+ \rangle = 0. \tag{2.47}$$

Then V_2 is determined up to an element in the kernel of L:

$$V_2 = (0, 0, 1/b) + a\phi, \tag{2.48}$$

with a arbitrary. This arbitrariness is always encountered in such asymptotic expansions, since we can split an expression of a particular order into items for the next orders.

– *Order 3*:

$$-LV_3 = 2Q(\phi, V_2) + \alpha\phi. \tag{2.49}$$

We calculate $Q(\phi, V_2) = (0, -\frac{1}{2}b, 0)$, so that α is given by:

$$\alpha = -\frac{\langle 2Q(\phi, V_2), \phi^+ \rangle}{\langle \phi, \phi^+ \rangle} = \frac{\sigma}{b(1+\sigma)}. \tag{2.50}$$

To express the normal form in the vicinity of the bifurcation, we first calculate that the crossing real eigenvalue is:

$$\mu = \frac{\sigma(r-1)}{\sigma+1} + \mathrm{O}[(r-1)^2]. \tag{2.51}$$

The Landau equation for the pitchfork bifurcation of the Lorenz model reads:

$$\dot{A} = \frac{\sigma(r-1)}{\sigma+1}A - \frac{\sigma}{b(1+\sigma)}A^3. \tag{2.52}$$

The equilibrium state is $A_e = \sqrt{b(r-1)}$, so that U_e is expressed as:

$$\begin{aligned} U_e &= A_e\phi + A_e^2 V_2 + \mathrm{O}(A_e^3) \\ &= (a, a, r-1) + \mathrm{O}[(r-1)^3], \end{aligned} \tag{2.53}$$

with $a = \sqrt{b(r-1)}$. Surprisingly, in this example the lowest relevant order gives the exact solution.

2.4.4. Application to the Liénard oscillators
The critical control parameter is $\epsilon = 0$:

$$L = \mathrm{D}F_{\lambda_c}(0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}. \tag{2.54}$$

$$L\phi = \mathrm{i}\omega\phi, \qquad \text{with} \quad \phi = (1, \mathrm{i}\omega). \tag{2.55}$$

Since $N(U)$ is here triadic, we define a triadic form:

$$T(U_1, U_2, U_3) = [0, -\tfrac{1}{3}r\,(x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3) - p\,x_1x_2x_3], \tag{2.56}$$

with $N(U) = T(U, U, U)$.

We will need to calculate explicitly some of the coefficients of N:

$$\begin{aligned} U &= W\,\phi + \overline{W}\,\overline{\phi} + V_2^{20}W^2 + V_2^{11}W\overline{W} + V_2^{02}\overline{W}^2 + \mathrm{O}(|W|^3), \\ N(U) &= W^3T(\phi, \phi, \phi) + 3W^2\overline{W}T(\phi, \phi, \overline{\phi}) \\ &\qquad + \text{complex conjugated} + \mathrm{O}(|W|^4), \\ T(\phi, \phi, \overline{\phi}) &= (0, -\tfrac{1}{3}r\mathrm{i}\omega - p). \end{aligned} \tag{2.57}$$

For simplicity we choose the duality product instead of a Hermitian product, to solve the Fredholm alternatives:

$$L^+ = \begin{pmatrix} 0 & -\omega^2 \\ 1 & 0 \end{pmatrix}. \tag{2.58}$$

$$L^+\phi^+ = \mathrm{i}\omega\phi^+, \qquad \text{with} \quad \phi^+ = (\mathrm{i}\omega, 1). \tag{2.59}$$

We are ready to apply the KBM algorithm:

– *Order 2*:

$$\begin{aligned} &(2\mathrm{i}\omega - L)V_{20}^2 = 0, \\ &-LV_{11}^2 = 0, \\ &\text{complex conjugated equations,} \end{aligned} \tag{2.60}$$

leading to $V_2 = 0$, up to an arbitrary, real vector-combination of ϕ and $\overline{\phi}$.

– *Order 3*: We only need to consider:

$$(\mathrm{i}\omega - L)V_3^{21} = 3T(\phi, \phi, \overline{\phi}) + \alpha\phi, \tag{2.61}$$

leading to

$$\alpha = -\frac{\langle 3T(\phi,\phi,\overline{\phi}),\phi^+\rangle}{\langle \phi,\phi^+\rangle} = \frac{r}{2} - \frac{3ip}{2\omega}. \tag{2.62}$$

We see that if $r > 0$ the Hopf bifurcation is supercritical.

3. Generic instabilities of spatial systems

3.1. Stability of equilibrium in spatial systems

We use the word spatial systems for evolution partial differential equations of the kind of those encountered in hydrodynamical problems. For spatial systems phase space is a function space with infinite dimension. Let $X(x,t) \in \boldsymbol{R}^N$ be a solution, with $x \in \boldsymbol{R}^d$. The more general spatial system can be written as:

$$\begin{aligned} &F_\lambda[X(x,t), \partial_x, \partial_t, x, t] = 0 \\ &\quad + \text{boundary conditions} \\ &\quad + \text{initial conditions.} \end{aligned} \tag{3.1}$$

When the space dimension, d, is greater than one, the symbol ∂_x represents the space gradient. The example we will present to illustrate the concepts of destabilization of equilibrium states will be such that F_λ does not depend on x and t. Furthermore, F_λ will be invariant under fundamental symmetries: space and time translations and reflections. Our aim is to describe the generic instabilities of the equilibria of such systems. Note that these equilibria will themselves be invariant under some of the symmetries.

The stability of an equilibrium at X_λ is solved by linearization of the system of equations around X_λ. The difficulty is then how to calculate the spectrum of the functional linear operator, and determine which modes are destabilized. When the spectrum is continuous it is not correct to speak of eigenvalues. We will, however, still use this word instead of the word singularities, which would be appropriate.

We will present 2D Rayleigh–Bénard convection, spatially coupled chemical reactions, and 2D thermohaline convection as physical examples. A model equation will be used for a small classification of the spatial systems that admit the most current symmetries encountered in physics: space and time translational invariance and reflection invariance.

3.2. 2D Rayleigh–Bénard convection

We consider a fluid between two horizontal plates maintained at constant temper-

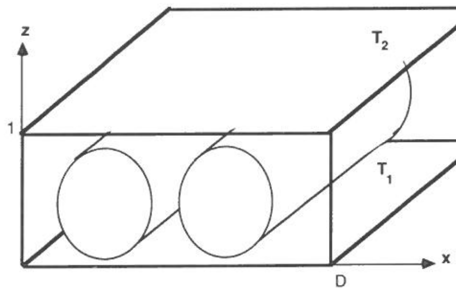

Fig. 5. Rayleigh–Bénard convection in a small box.

atures (fig. 5). We are interested in the 2D case:

$$\begin{aligned} \partial_t \Delta\psi &= \sigma\Delta^2\psi - R\sigma\,\partial_x\theta + J(\psi, \Delta\psi), \\ \partial_t\theta &= \Delta\theta - \partial_x\psi + J(\psi, \theta), \end{aligned} \tag{3.2}$$

where $\psi(x, z, t)$ is the stream function, from which one derives the horizontal velocity, $u = \partial_z\psi$, and the vertical velocity, $w = -\partial_x\psi$. $\theta(x, z, t)$ is the deviation of the temperature from the linear conductive profile. $J(f, g) = \partial_x f\, \partial_z g - \partial_z f\, \partial_x g$ is the Jacobian of f and g. The two control parameters are the Rayleigh number, R, and the Prandtl number, P.

Concerning the vertical boundary conditions, we assume free-slip conditions for the velocity and that the plates are perfect heat conductors:

$$\psi = 0, \quad \partial_x^2\psi = 0, \quad \theta = 0, \qquad \text{at} \quad z = 0, 1. \tag{3.3}$$

For the horizontal boundary conditions we distinguish two cases. In the small-box case we assume for simplicity horizontal periodicity of the fields. If D is the width of the box (in nondimensionalized units), the fundamental wave number is $k_0 = 2\pi/D$. The large-box case is the limit when D is very large.

3.2.1. Small box

Let us first assume that D is finite. We then perform a Fourier decomposition of the fields, taking into account the vertical boundary conditions:

$$U = \begin{pmatrix} \psi \\ \theta \end{pmatrix} = \sum_n \sum_k \widehat{U}_n(k, t)\, \mathrm{e}^{\mathrm{i}kx} \sin n\pi z, \tag{3.4}$$

with $n \in \boldsymbol{N}$, $k = mk_0$ and $m \in \boldsymbol{Z}$.

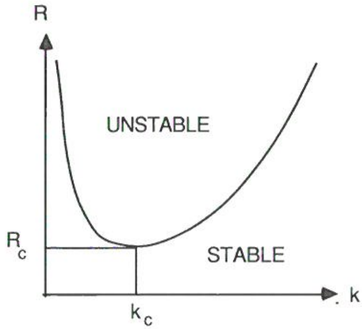

Fig. 6. Stability diagram for the Rayleigh–Bénard convection problem.

The equations become:

$$\partial_t \widehat{U}_n(k,t) = \widehat{L}_\lambda(n,k)\, \widehat{U}_n(k,t) + N_\lambda(U). \tag{3.5}$$

We note that the linear operator for the stability of the conductive state ($\psi = 0$, $\theta = 0$) is block diagonal in spectral space. For a given horizontal wave number k and a given n, a single block is a 2×2 matrix:

$$\widehat{L}_\lambda(n,k) = \begin{pmatrix} -\sigma q_n^2 & R\sigma \mathrm{i}k/q_n^2 \\ -\mathrm{i}k & -q_n^2 \end{pmatrix}, \tag{3.6}$$

with $q_n^2 = k^2 + n^2\pi^2$. The eigenvalues are given by the characteristic equation:

$$\begin{aligned} & s^2 + s(\sigma+1)q_n^2 + \sigma q_n^4 - R\sigma \frac{k^2}{q_n^2} = 0 \\ \Longleftrightarrow \quad & s^2 + s(\sigma+1)q_n^2 + \frac{\sigma k^2}{q_n^2}\left(\frac{q_n^6}{k^2} - R\right) = 0. \end{aligned} \tag{3.7}$$

For each n we can draw the curve $R_n(k) = q_n^6/k^2$ in a (k, R) plane, corresponding to the crossing of the imaginary axis by a real eigenvalue. The onset of instability for the conductive state occurs at $n = 1$. The minimum of the curve $R_1(k)$ gives the value of the critical Rayleigh number $R_c = 27\pi^4/4$ and the critical wave number $k_c = \pi/\sqrt{2}$ (fig. 6).

We now assume that $k_0 = k_c$, and try to derive the normal form of the instability at $R = R_c$. We use the notation $L = \widehat{L}_{\lambda_c}(1, k_c)$, and look at the kernels of the two singular blocks of L and $\overline{L}$, the two singular blocks of the big block diagonal operator. Their kernels can be calculated:

$$\begin{aligned} & \phi = (1, -\mathrm{i}k_c/q_c^2), \\ & L\phi = 0 \quad \text{and} \quad \overline{L}\,\overline{\phi} = 0, \end{aligned} \tag{3.8}$$

with $q_c^2 = k_c^2 + \pi^2$. We suppose that the marginal modes ϕ and $\overline{\phi}$ are governing the dynamics at the onset of instability, while the other modes follow them adiabatically:

$$U(x,z,t) = [W(t)\,\phi\,e^{ik_c x} + \overline{W}(t)\,\overline{\phi}\,e^{-ik_c x}]\sin\pi z + V[W(t),\overline{W}(t),x,z]. \tag{3.9}$$

To derive a priori the shape of the normal form, we use the symmetries of the Boussinesq equations. The translational invariance $[x \to x+h,\ \theta \to \theta,\ \psi \to \psi]$ implies $[W(t) \to W(t)e^{i\varphi}]$, and the reflection symmetry $[x \to -x,\ y \to -y,\ \theta \to -\theta,\ \psi \to \psi]$ implies $[W(t) \to \overline{W}(t)]$. The amplitude equation then reads:

$$\dot{W} = \mu W - \alpha |W|^2 W, \qquad \text{with} \quad \alpha \in \boldsymbol{R}, \tag{3.10}$$

where $\mu = (R - R_c)q_c^2\sigma/(\sigma+1)R_c + O(|R - R_c|^2)$ is the value of the critical eigenvalue when R is slightly above R_c. One may, e.g., calculate the nonlinear coefficient α with the KBM method, discussed in the section 2. This nonlinear coefficient happens to be positive for Rayleigh–Bénard convection: $\alpha = k_c^2\sigma/8(\sigma+1)$. If we set $W = A\exp(i\varphi)$, we get:

$$\begin{aligned} \dot{A} &= \mu A - \alpha A^3, \\ \dot{\varphi} &= 0. \end{aligned} \tag{3.11}$$

The same analysis can be done with any fundamental wave number k_0, and the new values of the critical Rayleigh number, μ, and α may be calculated explicitly.

3.2.2. Large box

When k_0 is very small, meaning that the box size, D, is very large (fig. 7), a quasi-continuum of modes is destabilized near k_c. Let $s(\lambda, k)$ be the destabilizing eigenvalue, given by the above characteristic equation as a function of the control parameters λ and k. In the vicinity of the critical parameter R_c and k_c we can expand s:

$$\begin{aligned} &s(\lambda,k) = \mu - \beta\,(k - k_c)^2, \\ &\quad \text{with} \quad \mu = \frac{\sigma k_c^2}{(\sigma+1)q_c^2}(R - R_c) \quad \text{and} \quad \beta = 2\frac{\partial^2 s}{\partial k^2}(\lambda_c, k_c). \end{aligned} \tag{3.12}$$

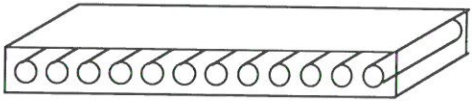

Fig. 7. Rayleigh–Bénard convection in a large box.

To derive the amplitude equation in the case of a large box, we let the amplitude vary with x, as for wave packets:

$$U(x,z,t) = [W(x,t)\,\phi\,\mathrm{e}^{\mathrm{i}k_c x} + \overline{W}(x,t)\,\overline{\phi}\,\mathrm{e}^{-\mathrm{i}k_c x}]\sin\pi z \\ + V[W(x,t), \overline{W}(x,t), x, z]. \tag{3.13}$$

We can calculate the effect of the total linear operator L in physical space on such a wave packet by noting that, provided $W(x,t)$ is slowly varying with space:

$$L_\lambda[W(x,t)\,\phi\,\mathrm{e}^{\mathrm{i}k_c x}\sin(\pi z)] = (\mu + \beta\,\partial_x^2)[W(x,t)\,\phi\,\mathrm{e}^{\mathrm{i}k_c x}\sin(\pi z)]. \tag{3.14}$$

This action of the linear operator on the wave packet is the root of the mathematical difficulty to justify the large box analysis. We must admit these asymptotic expansions, where the slow variation of the amplitude is the small parameter, in the sence of asymptotic methods like multiple scale analysis.

The amplitude equation finally reads:

$$\partial_t W = \mu W + \beta\,\partial_x^2 W - \alpha |W|^2 W, \tag{3.15}$$

where α and β are real. This is the Landau equation for a large box.

3.2.3. *Family of solutions*

This amplitude equation admits a family of solutions parametrized by a wave number Q:

$$W_Q(x) = A(Q)\,\mathrm{e}^{\mathrm{i}Qx+\mathrm{i}\varphi}, \\ A(Q) = \sqrt{\frac{\mu - \beta Q^2}{\alpha}}. \tag{3.16}$$

These solutions correspond to all the periodic structures for waves numbers k between $k_c - Q_{max}$ and $k_c + Q_{max}$, with $Q_{max} = \sqrt{\mu/\beta}$. We will see in section 4 that these structures are unstable for $|Q| > \sqrt{\mu/3\beta}$. We will interpret this so-called Eckhaus instability (fig. 8) physically with the help of phase theory.

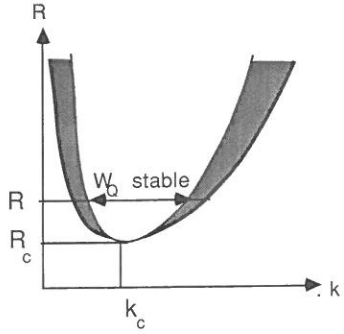

Fig. 8. Eckhaus instability for the Rayleigh–Bénard convection problem.

3.3. Spatially coupled oscillators

Let us first consider a chemical reaction whose kinetics is governed by the dynamical system:

$$\dot{X} = F_\lambda(X), \qquad \text{with} \quad X(t) \in \boldsymbol{R}, \tag{3.17}$$

where $X(t)$ is the vector of the chemical concentrations in the reaction. We suppose that an equilibrium is destabilized by two complex conjugated eigenvalues. If $U(t)$ is the perturbation around this equilibrium, the system becomes:

$$\dot{U} = M_\lambda U + N_\lambda(U). \tag{3.18}$$

The normal form of this Hopf bifurcation is obtained by considering M, the linear operator at the critical value λ_c, and its two complex eigenvectors ϕ and $\overline{\phi}$. We assume as usual:

$$U = W(t)\phi + \overline{W}(t)\overline{\phi} + V[W(t), \overline{W}(t)], \tag{3.19}$$

to derive the normal form:

$$\dot{W} = (\mu + \mathrm{i}\omega)W - \alpha|W|^2 W, \qquad \text{with} \quad \alpha \in \boldsymbol{C}. \tag{3.20}$$

3.3.1. Spatial coupling due to diffusion

The previous dynamical system is valid to describe the kinetics of a stirred medium. When the species are not stirred, spatial gradients become important and one must take into account the molecular diffusion. We are led to the partial differential system:

$$\partial_t X = F_\lambda(X) + B\,\partial_x^2 X, \qquad \text{with} \quad X(x,t) \in \boldsymbol{R}^N. \tag{3.21}$$

The $N \times N$ diffusion matrix B does not need to be diagonal, if cross diffusion is present. We consider for simplicity the 1D case $x \in \boldsymbol{R}$.

3.3.2. Small box

Let us consider periodic boundary conditions on the interval $[0, D]$. The fundamental wave number is then:

$$k_0 = 2\pi/D. \tag{3.22}$$

We can perform a Fourier transform of the perturbation, $U(x,t)$:

$$U(x,t) = \sum_k \widehat{U}(k,t)\,\mathrm{e}^{ikx}, \qquad \text{with} \quad k = mk_0, \quad m \in \boldsymbol{Z}. \tag{3.23}$$

The linear operator governing the stability of the equilibrium state reads:

$$\widehat{L}_\lambda = M_\lambda - k^2 \widehat{B}. \tag{3.24}$$

If the box is small we still observe a Hopf bifurcation described by the usual normal form.

3.3.3. Large box

If k_0 goes to 0 we must consider that a continuum of complex conjugated eigenvalues is crossing the imaginary axis. We express these bands of eigenvalues as a function of λ and k:

$$s(\lambda, k) = (\mu + \mathrm{i}\omega) + \beta k^2 + \mathrm{O}(|\lambda - \lambda_\mathrm{c}|^2, |k|^3), \tag{3.25}$$

where β is a complex coefficient. By assuming the following expression for $U(x,t)$:

$$U(x,t) = W(x,t)\,\phi + \overline{W}(x,t)\,\overline{\phi} + V[W(x,t), \overline{W}(x,t)], \tag{3.26}$$

we obtain the amplitude equation:

$$\partial_t W = (\mu + \mathrm{i}\omega)W + \beta\,\partial_x^2 W - \alpha |W|^2 W, \tag{3.27}$$

where $\alpha = \alpha_r + i\alpha_i$ and $\beta = \beta_r + i\beta_i$ are complex coefficients. This is the large-box version of the Landau–Ginzburg equation.

This equation admits a family of periodic solutions:

$$\begin{aligned} W_Q &= A(Q)\exp[i\Omega(Q)t + iQx + i\varphi], \\ A(Q) &= \left(\frac{\mu - \beta_r Q^2}{\alpha_r}\right), \\ \Omega(Q) &= \omega - \beta_i Q^2 - \alpha_i A^2. \end{aligned} \tag{3.28}$$

We will see in section 4 that these periodic solutions may be unstable depending on the values of Q and the coefficients. The most stable structure, W_0, is destabilized when $\alpha_r\beta_r + \alpha_i\beta_i < 0$.

3.4. Simple model for a classification

We consider the following model of spatial systems:

$$\partial_t U = L_\lambda(\partial_x)U + N_\lambda(U), \tag{3.29}$$

where $U(x,t) \in \boldsymbol{R}$ is periodic in x on the interval $[0, D]$. We perform a Fourier decomposition of $U(x,t)$ for the spatial variable. The fundamental wave number is $k_0 = 2\pi/D$:

$$U(x,t) = \sum_k \widehat{U}(k,t)\,e^{ikx}, \qquad \text{with} \quad k = mk_0, \quad m \in \boldsymbol{Z}. \tag{3.30}$$

We assume that L_λ is $N \times N$ block diagonal in Fourier space. Let $\widehat{L}_\lambda(k)$ denote the $N \times N$ block associated with the wave number k. L_λ is a collection of such blocks for all the possible wave numbers:

$$L_\lambda = [\,\ldots, \widehat{L}_\lambda(-2k_0), \widehat{L}_\lambda(-k_0), \widehat{L}_\lambda(0), \widehat{L}_\lambda(k_0), \widehat{L}_\lambda(2k_0), \ldots\,]. \tag{3.31}$$

Since L_λ is real in physical space, we have the following properties of the Fourier blocks:

$$\begin{aligned} &\widehat{L}_\lambda(-k) = \overline{\widehat{L}_\lambda}(k), \\ &\widehat{L}_\lambda(k)\,\phi_i(k) = s_i(k)\,\phi_i(k), \\ &\phi_i(-k) = \overline{\phi}_i(k) \quad \text{and} \quad s_i(-k) = \overline{s}_i(k). \end{aligned} \tag{3.32}$$

We assume the following properties of our simple model: space translational invariance $[x \to x+h,\ U \to U]$, time translational invariance $[t \to t+\tau,\ U \to U]$,

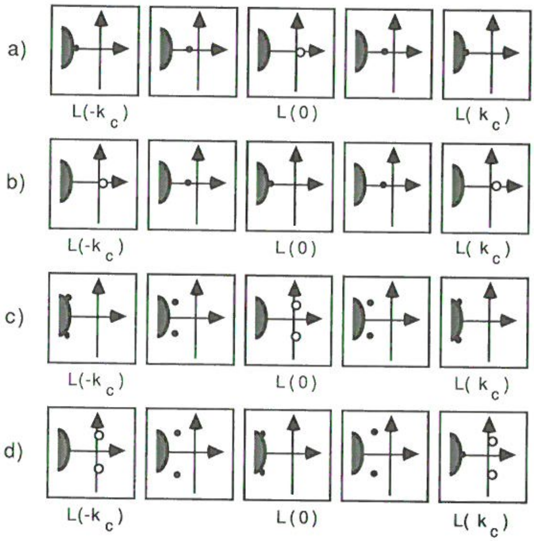

Fig. 9. The four generic cases of instability for a spatial system: (a) Real eigenvalue destabilizing $L_\lambda(0)$. (b) Real eigenvalue destabilizing $L_\lambda(k_c)$ and $L_\lambda(-k_c)$. (c) Complex eigenvalues destabilizing $L_\lambda(0)$. (d) Complex eigenvalues destabilizing $L_\lambda(k_c)$ and $L_\lambda(-k_c)$.

and reflection invariance $[x \to -x,\ U \to S(U)]$, where S is a symmetry we will assume to be the identity here.

We now investigate the destabilizations of the equilibrium state 0.

3.4.1. Four generic destabilizations

We consider that one of the $\widehat{L}_\lambda$ blocks is destabilizing at λ_c, and that $\widehat{L}_\lambda(k)$ is a differentiable function of k. Given the symmetries, we have assumed that there are four generic cases (fig. 9):

- Real eigenvalue destabilizing $\widehat{L}_\lambda(0)$.
- Real eigenvalue destabilizing $\widehat{L}_\lambda(k_c)$ and $\widehat{L}_\lambda(-k_c)$.
- Complex conjugated eigenvalues destabilizing $\widehat{L}_\lambda(0)$.
- Complex conjugated eigenvalues destabilizing $\widehat{L}_\lambda(k_c)$ and $\widehat{L}_\lambda(-k_c)$.

For the last case both $\widehat{L}_\lambda(k_c)$ and $\widehat{L}_\lambda(-k_c)$ must admit these two conjugated eigenvalues $\mathrm{i}\omega$ and $-\mathrm{i}\omega$ because of reflection symmetry. An example where this reflection symmetry is not present is:

$$\partial_t U = c\,\partial_x U, \tag{3.33}$$

with $\widehat{L}_\lambda(k) = \mathrm{i}ck$. In this example each block admits only one of the complex eigenvalues $\pm\mathrm{i}ck$. Less trivial examples of systems without reflection symmetry may exhibit the same property: one diagonal block $\widehat{L}_\lambda(k_c)$ admits one of two

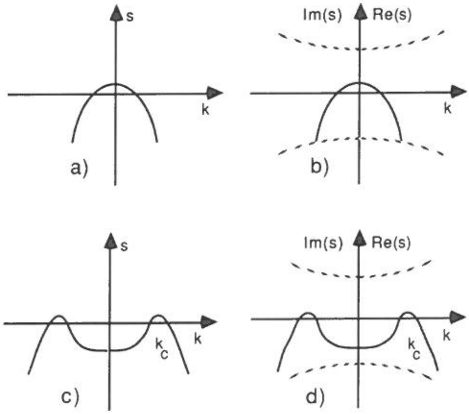

Fig. 10. Destabilizing continuum of eigenvalues for the large-box case: (a) Real eigenvalue destabilizing $L_\lambda(0)$. (b) Real eigenvalue destabilizing $L_\lambda(k_c)$ and $L_\lambda(-k_c)$. (c) Complex eigenvalues destabilizing $L_\lambda(0)$. (d) Complex eigenvalues destabilizing $L_\lambda(k_c)$ and $L_\lambda(-k_c)$.

conjugated eigenvalues, while $\widehat{L}_\lambda(-k_c)$ admits the other one. In that case the instability would be described by the Ginzburg–Landau equation in the reference frame moving with the group velocity of the wave packet. But with the reflection symmetry we have assumed we will see below that we need two coupled Ginzburg–Landau equations.

We now want to write down the amplitude equations for the above four cases, in the limit $k_0 \to 0$. The small-box case is recovered by omitting the spatial derivations. In each case one or two continua of modes will be responsible for the destabilization (fig. 10), and will be expressed as:

$$\begin{aligned} s(\lambda, k) &= s(\lambda_c, k_c) + \frac{\partial s}{\partial \lambda}(\lambda_c, k_c)(\lambda - \lambda_c) + \frac{\partial s}{\partial k}(\lambda_c, k_c)(k - k_c) \\ &\quad + \frac{1}{2}\frac{\partial^2 s}{\partial k^2}(\lambda_c, k_c)(k - k_c)^2 + \cdots \\ &= (\mu + \mathrm{i}\omega) + \mathrm{i}d(k - k_c) - \beta(k - k_c)^2 + \cdots . \end{aligned} \tag{3.34}$$

In the case of real destabilizing eigenvalues, ω is zero and β is real. In all cases d is real, because of reflection symmetry. It is the group velocity of the wave packet.

3.4.2. *Real eigenvalue destabilizing $\widetilde{L}_\lambda(0)$*

In this case the destabilizing continuum of eigenvalues is:

$$s(\lambda, k) = \mu - \beta k^2, \tag{3.35}$$

with β real. Let ϕ be the eigenmode of $\widehat{L}_\lambda(0)$:

$$U(x,t) = [W(x,t) + \overline{W}(x,t)]\phi + V(W, \overline{W}). \tag{3.36}$$

The amplitude equation must admit the symmetry $[W \to W\,\mathrm{e}^{\mathrm{i}\varphi}]$ and then reads:

$$\partial_t W = \mu W + \beta\,\partial_x^2 W - \alpha|W|^2 W, \tag{3.37}$$

where α and β are real coefficients.

3.4.3. Real eigenvalue destabilizing $\widetilde{L}_\lambda(k_\mathrm{c})$ and $\widetilde{L}_\lambda(-k_\mathrm{c})$
In this case the destabilizing continuum of eigenvalues is:

$$s(\lambda, k) = \mu - \beta(k - k_\mathrm{c})^2, \tag{3.38}$$

with β real. Let ϕ and $\overline{\phi}$ be, respectively, the eigenmodes of $\widehat{L}_\lambda(k_\mathrm{c})$ and $\widehat{L}_\lambda(-k_\mathrm{c})$:

$$U(x,t) = W(x,t)\,\phi\,\mathrm{e}^{\mathrm{i}k_\mathrm{c}x} + \overline{W}(x,t)\,\overline{\phi}\,\mathrm{e}^{-\mathrm{i}k_\mathrm{c}x} + V(W, \overline{W}). \tag{3.39}$$

The amplitude equation must admit the symmetry $[W \to W\,\mathrm{e}^{\mathrm{i}\varphi}]$ and then reads:

$$\partial_t W = \mu W + \beta\,\partial_x^2 W - \alpha|W|^2 W, \tag{3.40}$$

where α and β are real coefficients.

3.4.4. Complex eigenvalues destabilizing $\widetilde{L}_\lambda(0)$
In this case the destabilizing continuum of eigenvalues is:

$$s(\lambda, k) = (\mu + \mathrm{i}\omega) - \beta(k - k_\mathrm{c})^2. \tag{3.41}$$

Let ϕ and $\overline{\phi}$ be the eigenmodes of $\widehat{L}_\lambda(0)$:

$$U(x,t) = W(x,t)\,\phi + \overline{W}(x,t)\,\overline{\phi} + V(W, \overline{W}). \tag{3.42}$$

The amplitude equation must admit the symmetry $[x \to -x,\ W \to \overline{W}]$ and therefore reads:

$$\partial_t W = (\mu + \mathrm{i}\omega)W + \beta\,\partial_x^2 W - \alpha|W|^2 W, \tag{3.43}$$

where α and β are complex coefficients.

3.4.5. Complex eigenvalues destabilizing $\widetilde{L}_\lambda(k_c)$ and $\widetilde{L}_\lambda(-k_c)$

In this case the destabilizing continuum of eigenvalues is:

$$s(\lambda, k) = (\mu + \mathrm{i}\omega) + \mathrm{i}d(k - k_c) - \beta(k - k_c)^2, \tag{3.44}$$

with β complex and d real. Let ϕ_+, ϕ_-, $\overline{\phi}_+$, and $\overline{\phi}_-$ be the eigenmodes of $\widehat{L}_\lambda(k_c)$ and $\widehat{L}_\lambda(-k_c)$:

$$\begin{aligned} \widehat{L}_\lambda(k_c)\phi_+ &= \mathrm{i}\omega\phi_+, & \widehat{L}_\lambda(-k_c)\overline{\phi}_+ &= -\mathrm{i}\omega\overline{\phi}_+, \\ \widehat{L}_\lambda(k_c)\phi_- &= -\mathrm{i}\omega\phi_-, & \widehat{L}_\lambda(-k_c)\overline{\phi}_- &= \mathrm{i}\omega\overline{\phi}_-. \end{aligned} \tag{3.45}$$

We now assume the appropriate decomposition:

$$\begin{aligned} U(x,t) = {} & [W_+(x,t)\phi_+ + W_-(x,t)\phi_-]\, \mathrm{e}^{\mathrm{i}k_c x} \\ & + [\overline{W}_+(x,t)\overline{\phi}_+ + \overline{W}_-(x,t)\overline{\phi}_-]\, \mathrm{e}^{-\mathrm{i}k_c x} \\ & + V(W_+, W_-, \overline{W}_+, \overline{W}_-). \end{aligned} \tag{3.46}$$

The amplitude equation must admit the symmetries $[x \to -x, W_+ \to \overline{W}_-, W_- \to \overline{W}_+]$ and therefore reads:

$$\begin{aligned} \partial_t W_+ = {} & (\mu + \mathrm{i}\omega)W_+ + d\,\partial_x W_+ + \beta\,\partial_x^2 W_+ \\ & - \alpha|W_+|^2 W_+ - (\gamma + \mathrm{i}\delta)|W_-|^2 W_+, \\ \partial_t W_- = {} & (\mu + \mathrm{i}\omega)W_- - d\,\partial_x W_- + \overline{\beta}\,\partial_x^2 W_- \\ & - \overline{\alpha}|W_-|^2 W_- - (\gamma - \mathrm{i}\delta)|W_+|^2 W_-, \end{aligned} \tag{3.47}$$

where α and β are complex coefficients.

This system of two coupled Landau–Ginzburg equations admits two families of periodic solutions. In the small-box case there is no spatial dependency. The moduli of the complex variables $W_+(t)$ and $W_-(t)$ are decoupled from the phases and form a Volterra system. This Volterra system admits four fixed points: the null state, two fixed points such that one of the amplitudes is zero (they correspond to a left or a right travelling wave), and one fixed point where the two amplitudes are equal (this corresponds to a standing wave). The stability of the four states depends on the coefficients, and we leave this discussion as an exercise. In the large-box case, each state is replaced by a family of periodic structures that one can express analytically.

3.4.6. Summary and supercritical cases

Our classification of the generic destabilizations of spatial systems possessing translational and reflection symmetry leads to three basic amplitude equations.

By rescaling the space and time variables and the amplitudes, we can write these amplitude equations with the minimum of arbitrary coefficients. We write below these rescaled equations in the supercritical cases, the subcritical case being obtained by changing the sign of the real part of the nonlinear coefficients.

The Landau equation:

$$W_t = W + W_{xx} - |W|^2 W. \tag{3.48}$$

The Landau–Ginzburg equation:

$$W_t = (1 + \mathrm{i}c_0)W + (1 + \mathrm{i}c_1)W_{xx} - (1 + \mathrm{i}c_2)|W|^2 W. \tag{3.49}$$

The Volterra coupled Landau–Ginzburg equations:

$$\begin{aligned}\partial_t W_+ - \partial_x W_+ &= (1 + \mathrm{i}c_0)W_+ + (1 + \mathrm{i}c_1)\partial_x^2 W_+ \\ &\quad - (1 + \mathrm{i}c_2)|W_+|^2 W_+ - (\gamma + \mathrm{i}\delta)|W_-|^2 W_+, \\ \partial_t W_- + \partial_x W_- &= (1 - \mathrm{i}c_0)W_- + (1 - \mathrm{i}c_1)\partial_x^2 W_+ \\ &\quad - (1 - \mathrm{i}c_2)|W_-|^2 W_- - (\gamma - \mathrm{i}\delta)|W_+|^2 W_-.\end{aligned} \tag{3.50}$$

Each of the three amplitude equations admits one or several families of solutions with different spatial structure, parametrized by a wave number Q. We will discuss the stability of these solutions in section 4 and use the phase equation theory to describe the dynamics of the newly encountered instabilities.

3.5. Thermohaline convection

As for simple Rayleigh–Bénard convection, we consider a fluid between two horizontal plates maintained at constant temperatures, but we add the effect of a linear stratification, e.g., due to salinity. We are still interested in the 2D case:

$$\begin{aligned}\partial_t \Delta\psi &= \sigma\Delta^2\psi - R\sigma\,\partial_x\theta + S\sigma\tau\,\partial_x\Sigma + J(\psi, \Delta\psi), \\ \partial_t\theta &= \Delta\theta - \partial_x\psi + J(\psi, \theta), \\ \partial_t\Sigma &= \tau\Delta\Sigma + \partial_x\psi + J(\psi, \Sigma),\end{aligned} \tag{3.51}$$

where Σ is the deviation of the salinity from the linear profile. In addition to the two control parameters R and P, there is the saline Rayleigh number, S.

We choose simple boundary conditions of the free-slip type:

$$\psi = 0, \quad \partial_x^2\psi = 0, \quad \theta = 0, \quad \Sigma = 0, \qquad \text{at} \quad z = 0, 1. \tag{3.52}$$

In the horizontal direction, we assume periodic boundary conditions with fundamental wave number $k_0 = 2\pi/D$, and we are interested again in the small-box case or the large-box case, $k_0 \to \infty$.

The linear stability depends on the matrix:

$$\widehat{L}_\lambda(n,k) = \begin{pmatrix} -\sigma q_n^2 & R\sigma \mathrm{i}k/q_n^2 & -S\sigma\tau \mathrm{i}k/q_n^2 \\ -\mathrm{i}k & -q_n^2 & 0 \\ \mathrm{i}k & 0 & -\tau q_n^2 \end{pmatrix}, \tag{3.53}$$

with $q_n^2 = k^2 + n^2\pi^2$. The eigenvalues are given by the characteristic equation:

$$(s+\tau q_n^2)\left[s^2 + s(\sigma+1)q_n^2 + \frac{\sigma k^2}{q_n^2}\left(\frac{q_n^6}{k^2} - R\right)\right] + S\sigma\tau k^2 = 0. \tag{3.54}$$

We assume that instability occurs for the modes $n = 1$. There is a curve in the (R, S) plane (a codimension-one surface in parameter space) for which the instability is due to two complex conjugated eigenvalues at finite wave number k_c. The reflection symmetry of the Boussinesq equation implies that both $L_\lambda(1, k_c)$ and $L_\lambda(1, -k_c)$ admit these two conjugated eigenvalues.

The amplitude equations are similar to those described in the previous section: two coupled Ginzburg–Landau equations. But when one carries out the calculation, the coefficient α turns out to be purely imaginary. The nondimensional version of the coupled amplitude equation we have given, is no longer valid and one must read $\mathrm{i}c_2$ instead of $1 + \mathrm{i}c_2$. The discussion of the existence of a family of travelling or standing waves is slightly modified.

4. Introduction to the phase equation theory

We have seen in the previous sections that the dynamics of the most common hydrodynamical instabilities is contained in the two following equations (after suitable scaling of space and time variables and for the supercritical cases):

$$\begin{aligned} &\text{the Landau equation:} \quad W_t = W + W_{xx} - |W|^2 W, \\ &\text{the Landau–Ginzburg equation:} \\ &\qquad W_t = (1+\mathrm{i}c_0)W + (1+\mathrm{i}c_1)W_{xx} - (1+\mathrm{i}c_2)|W|^2 W. \end{aligned} \tag{4.1}$$

These amplitude equations admit families of periodic structures given by:

$$\begin{aligned} &W_Q(x,t) = A(Q)\,\mathrm{e}^{\mathrm{i}\Omega(Q)t + \mathrm{i}Qx + \mathrm{i}\varphi}, \\ &\text{with} \quad A(Q) = (1-Q^2)^{1/2} \\ &\text{and} \quad \Omega(Q) = c_0 + (c_2 - c_1)Q^2, \end{aligned} \tag{4.2}$$

where the Landau-equation case is obtained by setting the c_i's equal to 0. φ is an arbitrary phase, which arises from the invariance of the equation under space translation.

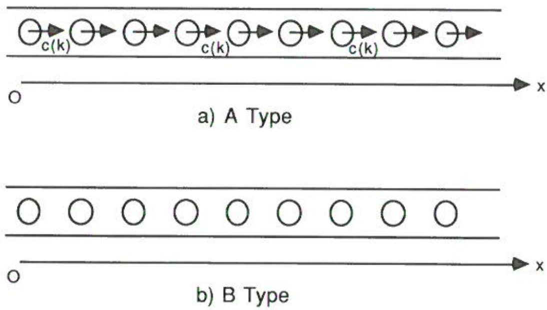

Fig. 11. A-type (dispersive) or B-type (nondispersive) structures: (a) A type, (b) B type.

We wish now to make some remarks about families of periodic structures in general, and solutions of equations due to certain symmetries.

4.1. *Family of periodic structures*

4.1.1. *A-type or B-type periodic structures*

Let

$$\partial_t X = F(X, \partial_x, \partial_y), \qquad \text{with} \quad X(x, y, t) \in \boldsymbol{R}^N \tag{4.3}$$

be invariant under:

- space translation $[x \to x + h,\ X \to X]$,
- time translations $[t \to t + \tau,\ X \to X]$,
- space reflection $[x \to -x,\ X \to X]$.

A family of 1D periodic structures may be of the A or B type:

$$\begin{aligned} &\text{A type:} \quad X(x, y, t) = X_k[x + c(k)t + h], \\ &\text{B type:} \quad X(x, y, t) = X_k[x + h], \end{aligned} \tag{4.4}$$

with h arbitrary.

The A-type structures are "dispersive", while B-type ones are not (fig. 11).

4.1.2. *Marginal mode*

The system equations (4.1) being invariant under space translation, one can show that each periodic structure admits a marginal mode, in the study of its stability.

Let $X_k(x)$ be a B-type structure (the same arguments hold for the A type). Then $X(x + h)$ is a new solution, induced by translation. If that translation is small, we write:

$$X_k(x + h) = X_k(x) + hu_0 + \mathrm{O}(h^2), \tag{4.5}$$

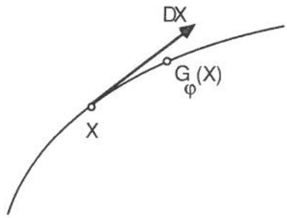

Fig. 12. Marginal mode tangent to a group orbit.

with $u_0 = \partial_x X_k(x)$.

To study the stability of $X_k(x)$, we set $X(x, y, t) = X_k(x) + u(x, y, t)$. Putting this expression into eq. (4.1) and keeping only the linear parts, one has to solve:

$$\partial_t u = Lu. \tag{4.6}$$

The stability problem is solved by looking at the spectrum of L.

A small perturbation proportional to u_0 is neither amplified nor damped. The mode u_0 is a marginal mode and satisfies: $Lu_0 = 0$.

More generally, let us suppose that the system (4.1) is invariant under the action of a continuous group G_φ:

$$\begin{aligned} &G_\varphi : \; X \to G_\varphi X, \\ &G_{\varphi_1} \circ \; G_{\varphi_2} = G_{\varphi_1+\varphi_2}, \\ &G_0 = I, \end{aligned} \tag{4.7}$$

where I stands for the identity operator. For example, in the translational case $G_h X = X(x + h)$.

If φ is small, we write:

$$G_\varphi X = \mathrm{e}^{\varphi D} X = X + \varphi DX + \tfrac{1}{2}\varphi^2 D^2 X + \mathrm{o}(\varphi^3), \tag{4.8}$$

where D is the generator of the group, defined by:

$$D = \lim_{\varphi \to 0} \frac{G_\varphi - I}{\varphi}. \tag{4.9}$$

In the translational case $D = \partial_x$. We claim that if X_k is a solution of a system invariant under the transformations in G_φ, then $u_0 = DX_k$ is a marginal mode for the stability of the solution X_k (fig. 12).

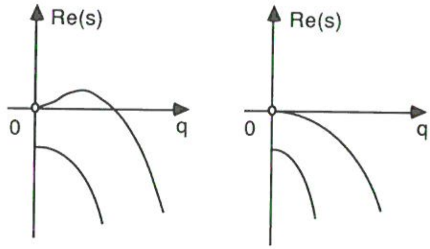

Fig. 13. The phase branch of quasi-marginal phase modes and a damped branch: (a) stable case, (b) phase instability.

4.1.3. *Family of marginal modes*

We have seen that u_0 is a marginal mode due to an invariance of the equations under a continuous symmetry group. In a large domain we can find continua of modes as well as isolated modes. Most of the time u_0 is not isolated, but belongs to a continuum of modes. The neighboring modes of u_0 are nearly marginal, and correspond to slow modulations in space of the phase. A perturbation of the basic solution X_k by these modes may approximatively be written as the following form:

$$X(x, y, t) = G_{\varphi(x,y,t)} X_k(x). \tag{4.10}$$

Speaking in terms of a spatial length scale parameter q for these modes (a Fourier wave number), the eigenvalues associated with that continuum of phase modes is a function $s(q)$, vanishing at $q = 0$. In the neighborhood of $q = 0$ the real part of s may be negative as well as positive. In the latter case one speaks of phase instability (fig. 13).

4.2. *Phase equation for the Landau equation*

We want to study the stability of the periodic structures W_Q of the Landau equation:

$$\begin{aligned} &W_t = (1 - |W|^2)W + W_{xx}, \\ &W_Q(x) = A(Q)\, e^{iQx + i\varphi}, \\ &\text{with} \quad A(Q) = (1 - Q^2)^{1/2}. \end{aligned} \tag{4.11}$$

These periodic structures are of the B type. We will see that they may encounter phase instability associated with the space invariance of the Landau equation.

4.2.1. Stability of the W_Q's

We can write a perturbation of W_Q in various ways:

$$\begin{aligned} &W(x,t) = W_Q(x) + u(x,t), \\ \text{or}\quad &W(x,t) = W_Q(x)[1 + n(x,t)]. \end{aligned} \tag{4.12}$$

We choose a way of expressing such a perturbation which is convenient for the action of the translational group:

$$W(x,t) = W_Q(x)\,\mathrm{e}^{\gamma(x,t)}, \tag{4.13}$$

with $\gamma(x,t) = \eta(x,t) + \mathrm{i}\psi(x,t)$.

Indeed, the action of the translational semi-group on W_Q corresponds to an imaginary constant γ. The action of the generator operator D on W_Q is multiplication by i.

To place this expression into the Landau equation some elementary algebra is needed. For instance, it is useful to use Leibnitz' formula for the derivation of:

$$\partial_x^2 W = (\partial_x^2 W_Q)\,\mathrm{e}^{\gamma} + 2(\partial_x W_Q)(\partial_x\,\mathrm{e}^{\gamma}) + W_Q(\partial_x^2\,\mathrm{e}^{\gamma}). \tag{4.14}$$

One finally obtains a system equivalent to the Landau equations:

$$\begin{aligned} \eta_t &= (1 - Q^2)(1 - \mathrm{e}^{2\eta}) + \eta_x^2 - \psi_x^2 + \eta_{xx} - 2Q\psi_x, \\ \psi_t &= 0 + 2\eta_x\psi_x + \psi_x x + 2Q\eta_x. \end{aligned} \tag{4.15}$$

Since we are interested in the stability of the basic state $(\eta, \psi) = 0$, we split in linear and nonlinear terms:

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = L(Q, \partial_x) \begin{pmatrix} \eta \\ \psi \end{pmatrix} + N(Q, \partial_x, \eta, \psi), \tag{4.16}$$

with

$$L(Q, \partial_x) = \begin{pmatrix} -2(1 - Q^2) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2Q \\ 2Q & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_x^2 \tag{4.17}$$

and

$$N(Q, \partial_x, \eta, \psi) = \begin{pmatrix} -(1 - Q^2)\sum_{p=2}^{\infty} 2\eta^p/p! + \eta_x^2 \\ 2\eta_x\psi_x \end{pmatrix}. \tag{4.18}$$

The spectrum of L is studied in Fourier space:

$$L(Q, \mathrm{i}q) = \begin{pmatrix} -2(1-Q^2) - q^2 & -2\mathrm{i}qQ \\ 2\mathrm{i}qQ & -q^2 \end{pmatrix}. \tag{4.19}$$

The eigenvalues are given by the characteristic equation:

$$s^2 - 2s[q^2 + (1-Q^2)] + [q^4 + 2q^2(1-3Q^2)] = 0. \tag{4.20}$$

We check that there is a marginal mode for $q = 0$ equal to $(\eta, \psi) = (0, 1)$ in the coordinate system that we have chosen. In the vicinity of $q = 0$ the branch phase reads:

$$s(q) = -\nu(Q)q^2 - \lambda(Q)q^4 + \mathrm{O}(q^6),$$
$$\text{with} \quad \nu(Q) = \frac{1-3Q^2}{1-Q^2} \quad \text{and} \quad \lambda(Q) = \frac{2Q^4}{(1-Q^2)^3}. \tag{4.21}$$

We see that for $|Q| < 3^{-1/2}$ the periodic structures W_Q are stable. For $3^{-1/2} < |Q| < 1$, they are unstable. The instability occurring at $|Q| = 3^{-1/2}$ is called the Eckhaus instability. Taking the example of Rayleigh–Bénard convection, the W_Q periodic structures describe all the possible 2D convection-roll structures at the onset of convection, with wave numbers in the vicinity of the critical one, k_c. Only a band around k_c of such structures is stable with respect to the Eckhaus instability (fig. 8).

4.2.2. *Derivation of the phase equation*

We now want to reduce this instability to the simplest dynamics, by eliminating the slaved modes. The expression for $L(Q, \mathrm{i}q)$ tells us that the branch of damped modes is issued from the homogeneous damped mode $(\eta, \psi) = (1, 0)$, associated with the eigenvalue $s = -2$. These modes correspond to a variation of the amplitude of W. The normal form we will obtain now by eliminating these amplitude modes, to renormalize the dynamics of the phase modes, will be called a phase equation.

Let us describe the spirit of such an elimination. We write that $\eta(x, t)$ is small and follows adiabatically the dynamics of the phase, $\psi(x, t)$: $\eta = N(\psi)$. Direct methods of elimination, as the BKM method, are difficult to apply in this case. We prefer to introduce small parameters related to the slow space and time variation of the phase. Practically, we say that η, ∂_t and ∂_x are small parameters. Various asymptotic justifications of the following calculation exist, we try here to give the shortest way to get the final result.

Keeping from the (η, ψ)-system (4.15) only the terms which are of relevant order, we reach as final result:

$$\begin{aligned} \eta_t &= -2\eta(1-Q^2) - 2\eta^2(1-Q^2) \\ &\quad - 2Q\psi_x + \eta_x^2 - \psi_x^2 + \eta_x x + \mathrm{O}(\eta^3), \\ \psi_t &= 2Q\eta_x + \psi_{xx} + 2\eta_x\psi_x. \end{aligned} \tag{4.22}$$

At *order* 1 in η we get:

$$\begin{aligned} \eta &= -\frac{Q}{(1-Q^2)}\psi_x, \\ \psi_t &= \frac{1-3Q^2}{1-Q^2}\psi_{xx} = \nu(Q)\psi_{xx}. \end{aligned} \tag{4.23}$$

We conclude that ∂_x is of order η, while ∂_t is of order η^2.

At *order* 2 in η we find:

$$\begin{aligned} \eta &= -\frac{Q}{(1-Q^2)}\psi_x - \frac{1+Q^2}{2(1-Q^2)^2}\psi_x^2, \\ \psi_t &= \frac{1-3Q^2}{1-Q^2}\psi_{xx} - \frac{4Q^2}{(1-Q^2)^2}\psi_x\psi_{xx} \\ &= \nu(Q)\psi_{xx} + g(Q)\psi_x\psi_{xx}. \end{aligned} \tag{4.24}$$

We note that $g(Q)$ is the derivative of $\nu(Q)$ to the parameter Q. We will see next that this coincidence is due to the consistency of the family of phase equations derived for the family of periodic structures.

When the linear coefficient of this phase equation is negative, we need to add the next linear term in order to obtain a well-posed partial differential equation. The final form of the phase equation at the minimum order reads:

$$\psi_t = \nu(Q)\psi_{xx} - \lambda(Q)\psi_{xxxx} + g(Q)\psi_x\psi_{xx}. \tag{4.25}$$

The Eckhaus instability is subcritical and the system evolves towards stable structures by nucleation. These processes may be analyzed with the Landau equation, with reference to the theory of defects (see ref. [11]). But the phase equation is no longer valid at the locations of the defects.

4.3. Phase equation for the Ginzburg–Landau equation

We now want to study the stability of the dispersive periodic structures W_Q of the Landau–Ginzburg equations:

$$\begin{aligned} & W_t = (1+\mathrm{i}c_2)(1-|W|^2)W + (1+\mathrm{i}c_1)W_{xx}, \\ & W_Q(x,t) = A(Q)\,\mathrm{e}^{\mathrm{i}\Omega(Q)t + \mathrm{i}Qx + \mathrm{i}\varphi}, \\ & \text{with} \quad A(Q) = (1-Q^2)^{1/2} \\ & \text{and} \quad \Omega(Q) = c_2 - c_1. \end{aligned} \tag{4.26}$$

Note that c_0 has been removed by the change of variable $W \to W\,\mathrm{e}^{\mathrm{i}c_0 t}$. These periodic structures are of the A type. We will concentrate our attention on the phase instability of the homogeneous oscillation $W_0(t)$. The stability of the other solutions W_Q may be discussed in the same spirit, and W_0 is found to be the most stable solution.

4.3.1. Stability of W_0

We use the same change of variable as the one we have used for the Landau equation, made appropriate for the stability study of W_0:

$$W(x,t) = W_0(x,t)\,\mathrm{e}^{\eta(x,t)+\mathrm{i}\psi(x,t)}. \tag{4.27}$$

The Landau–Ginzburg equation is then equivalent to:

$$\begin{aligned}\eta_t &= (1-\mathrm{e}^{2\eta}) + (\eta_{xx}+\eta_x^2-\psi_x^2) - c_1(\psi_{xx}-2\eta_x\psi_x),\\ \psi_t &= c_2(1-\mathrm{e}^{2\eta}) + c_1(\eta_{xx}+\eta_x^2-\psi_x^2) + (\psi_{xx}-2\eta_x\psi_x).\end{aligned} \tag{4.28}$$

The linear operator for the stability of W_0 here reads in Fourier space:

$$L(0,\mathrm{i}q) = \begin{pmatrix} -2-q^2 & c_1 q^2 \\ -2c_2 - c_1 q^2 & -k^2 \end{pmatrix}. \tag{4.29}$$

The eigenvalues of this operator are given by the characteristic equation:

$$s^2 + 2q^2 s + [(1+c_1^2)q^4 + 2(1+c_1c_2)q^2] = 0. \tag{4.30}$$

The phase branch, issued from the marginal mode $(\eta,\psi) = (1,0)$, can be expanded in a Taylor series around $q = 0$:

$$s(q) = -\nu q^2 - \lambda q^4 + \mathrm{O}(q^6), \tag{4.31}$$

with $\nu = 1 + c_1c_2$ and $\lambda = \frac{1}{2}c_1^2(1+c_2^2)$.

There is another way for calculating ν and λ, more rapidly than by performing a Taylor expansion of the characteristic equation. One expresses the 2×2 matrix $L(0,\mathrm{i}q) = A + q^2 B$ as a perturbation of $A = L(0,0)$. Let $U_0 = (0,1)$ and $U_1 = (1,c_2)$ be the eigenvectors of A, and $U_0^+ = (-c_2, 1)$ and $U_1^+ = (-c_2, 1)$ the eigenvectors of the transposed matrix A^+. A classical analysis of the eigenvalues of the perturbed operator gives:

$$\begin{aligned}\nu &= \frac{U_0^+ B U_0}{U_0^+ U_0} = 1 + c_1 c_2,\\ \lambda &= \frac{(U_0^+ B U_1)(U_1^+ B U_0)}{-2(U_0^+ U_0)(U_1^+ U_1)} = \frac{c_1^2(1+c_2^2)}{2}.\end{aligned} \tag{4.32}$$

If $\nu < 0$, the solution W_0 is unstable. The destabilizing modes correspond physically to phase changes slowly varying in space (fig. 14). The name "phase instability" associated with a symmetry group of the equations, comes from this particular example. In this case, both time and space translation invariance can be held responsible for the existence of a marginal mode.

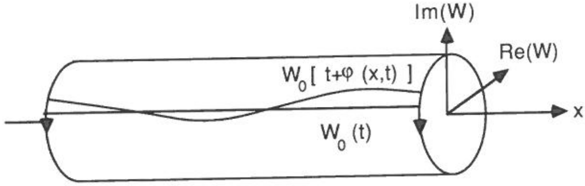

Fig. 14. Slow variation of the phase of W_0.

4.3.2. Phase equation derivation

As for the Landau equation, we can eliminate the slaved modes to obtain an equation for the phase dynamics. At the lowest order the dependence of η on the phase ψ, and the phase equation are found to be:

$$\begin{aligned} \eta(x,t) &= -\tfrac{1}{2}(c_1\psi_{xx} + \psi_x^2), \\ \psi_t &= \nu\,\psi_{xx} - \lambda\,\psi_{xxxx} + \mu\,\psi_x^2, \end{aligned} \tag{4.33}$$

with $\nu = 1 + c_1c_2$, $\lambda = \frac{1}{2}c_1^2(1+c_2^2)$, and $\mu = c_2 - c_1$.

This so-called Kuramoto–Tsuzuki equation [9] is found to contain complex, perhaps chaotic dynamics. Since the phase instability is supercritical, this phase equation is valid to represent the dynamics in the unstable case.

We notice that $\varphi(x,t) = \epsilon x + \mu\epsilon^2 t$ is a particular solution of this phase equation. But the corresponding perturbation $W(x,t) = W_0(t)\exp i\varphi(x,t)$ may be seen as the W_ϵ structure. By saying that $\Omega(\epsilon)$ must equal $\mu\epsilon^2$, one recovers immediately that $\mu = c_2 - c_1$. This argument will be generalized at the end of section 4.

4.4. Phase equations for periodic structures

We have derived explicitly the phase equation around the periodic structures, solutions of the Landau (B type) or Landau–Ginzburg equation (A type). Following ref. [10] we now show by symmetry and scaling arguments, that the shapes of these phase equations are quite general for any family of A- or B-type periodic structures.

4.4.1. B-type structures

Let $X_k(x)$ be a B-type family of periodic structures which are solutions of a 2D system of equations having space and time translational symmetry and space reflection symmetry, as described previously. We know that $\partial_x X_k$ is a marginal mode for the stability of X_k. This mode is due to the space translational invariance. A family of quasi-marginal modes is issued from this mode, and corresponds to slowly modulated translations in space. We then express any perturbation of X_k

in the following way:

$$X(x, y, t) = X_k \left[x + \psi(x, y, t) \right] + V \left[\psi(x, y, t) \right]. \tag{4.34}$$

The phase equation around the structure X_k is obtained by eliminating the slaved branches. One can derive the form of the terms of this phase equation by just symmetry arguments.

First, the reflection symmetry $[y \to -y,\ X \to X]$ implies the symmetry $[y \to -y,\ \psi \to \psi]$ for the phase equation. The order of ∂_y in the terms of the phase equation must be even, as for ψ_t. Thus the following terms are permitted: $\psi_{yy}, \psi_y^2, \psi_{4y}, \psi_{3y}\psi_y$, etc.

Second, the reflection symmetry $[x \to -x,\ X \to X]$ for B-type structures implies the symmetry $[x \to -x,\ \psi \to -\psi]$ for the phase equation. The only permitted terms are those for which the order of ∂_x plus the order of ψ is odd, as is the case for ψ_t. The following terms are permitted: $\psi_{xx}, \psi_{4x}, \psi_x\psi_{2y}$, etc.

The general form of the phase equation, at lowest order, is therefore:

$$\begin{aligned} \psi_t = {} & a_{20}\psi_{2x} + a_{02}\psi_{2y} + a_{40}\psi_{4y} + a_{22}\psi_{2x}\psi_{2y} + a_{04}\psi_{4y} \\ & + \text{other linear terms} + g\psi_x\psi_{2x} + g_1\psi_x\psi_{2y} + g_2\psi_y^2\psi_{2y} \\ & + \text{other nonlinear terms}. \end{aligned} \tag{4.35}$$

4.4.2. Eckhaus instability of B-type structures

This instability occurs when $a_{20} = -\epsilon$ becomes negative. We take this coefficient as a small parameter and assume the following scaling:

$$\partial_x \sim \epsilon^\nu \quad \text{and} \quad \psi \sim \epsilon^\beta.$$

The destabilizing term $-\epsilon\psi_{2x}$ must be of the order of the dominant dissipative term ψ_{4x}. We deduce that $\nu = \frac{1}{2}$, and the dominating nonlinear term is therefore $\psi_x\psi_{xx}$. The phase equation of the Eckhaus instability of B-type structures then reads, at the lowest order:

$$\psi_t = a_{20}\psi_{xx} + a_{40}\psi_{xxxx} + g\psi_x\psi_{xx}. \tag{4.36}$$

4.4.3. Zig-zag instability of B-type structures

When the destabilizing coefficient is $a_{02} = -\epsilon$, the instability is bidimensional and called the zig-zag instability (fig. 15). Both x and y dependencies are involved. We consider the scaling:

$$\partial_x \sim \epsilon^\nu, \quad \partial_y \sim \epsilon^{\nu'} \quad \text{and} \quad \psi \sim \epsilon^\beta. \tag{4.37}$$

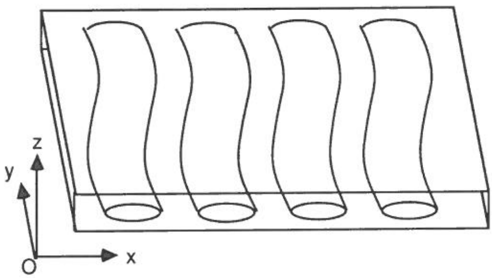

Fig. 15. Zig-zag instability of a periodic structure.

The destabilizing term $-\epsilon\psi_{2y}$ must be balanced by the two linear dissipative terms ψ_{4y} and ψ_{2x}, so that $\nu' = \frac{1}{2}$ and $\nu = 1$. The two dominant nonlinear terms are then $\psi_x\psi_{2y}$ and $\psi_y^2\psi_{2y}$. The phase equation for the zig-zag instability then reads, at the lowest order:

$$\psi_t = a_{02}\psi_{yy} + a_{04}\psi_{yyyy} + a_{20}\psi_{xx} + g_1\psi_x\psi_{yy} + g_2\psi_y^2\psi_{yy}. \tag{4.38}$$

4.4.4. A-type periodic structures

We assume now the existence of "dispersive" periodic solutions $X_k[x + c(k)t]$ of the 2D system of equations.

The reflection symmetry $[y \to -y,\ X \to X]$ leads to the same constraints on the form of the phase equation as for the B-type cases: the order in ∂_y must be even.

But the reflection symmetry $[x \to -x,\ X \to X]$ has no consequence on this shape. It only implies that a solution X_{-k} propagates with an opposite speed $c(-k) = -c(k)$.

The general form of phase equations for A-type structures allows more terms than in the B-type case:

$$\begin{aligned}\psi_t &= a_{10}\psi_x + a_{20}\psi_{2x} + a_{02}\psi_{2y} + a_{30}\psi_{3x} + a_{12}\psi_{xyy}\\ &\quad + a_{40}\psi_{4x} + a_{22}\psi_{2x}\psi_{2y} + a_{04}\psi_{4y} + \text{other linear terms}\\ &\quad + g_1\psi_x^2 + g_2\psi_y^2 + \text{other nonlinear terms}.\end{aligned} \tag{4.39}$$

4.4.5. Eckhaus instability of A-type structures

When $a_{20} = -\epsilon$ is negative, the X_k structure becomes unstable. Scaling analysis shows that both x and y dependencies are involved in the Eckhaus instability. The phase equation reads, at the lowest order:

$$\psi_t = a_{10}\psi_x + a_{20}\psi_{2x} + a_{30}\psi_{3x} + a_{40}\psi_{4x} + g_1\psi_x^2. \tag{4.40}$$

4.4.6. Zig-zag instability of A-type structures

When $a_{02} = -\epsilon$ is negative, the X_k structure becomes unstable. Scaling analysis shows that both x and y dependencies are involved in the zig-zag instability. The phase equation reads, at the lowest order:

$$\psi_t = a_{20}\psi_{2x} + a_{02}\psi_{2y} + a_{12}\psi_{xyy} + a_{04}\psi_{4y} + g_2\psi_y^2. \tag{4.41}$$

4.5. Family of phase equations

Around each B-type or A-type periodic structure it is possible to construct a phase equation. We can then consider the family of phase equations associated with the family of periodic structures parametrized by k. The linear coefficients of these phase equations are the Taylor expansion of the eigenvalues of the stability operators of the X_k's. We show how to derive the nonlinear coefficients as a function of these linear coefficients, by analyzing the consistency of the family of phase equations.

4.5.1. B-type Eckhaus nonlinear coefficients

The deduction of g as a function of the linear coefficient is obtained using the following argument. Let $\psi(x,t) = \epsilon x + \varphi(x,t)$, a change of variable for ψ, with an arbitrary small parameter ϵ. The phase equation for the B-type Eckhaus instability becomes:

$$\varphi_t = [a_{20}(k) + \epsilon g(k)]\varphi_{xx} + a_{40}\varphi_{4x} + g(k)\varphi_x\varphi_{xx}. \tag{4.42}$$

For ϵ small, the function φ may be seen as a phase perturbation of the periodic structure $X_{k(1+\epsilon)}$, since:

$$X_k(x+\epsilon) = X_{k(1+\epsilon)}(x) + \mathrm{o}(\epsilon^2). \tag{4.43}$$

This phase φ must obey the phase equation associated with the stability of $X_{k(1+\epsilon)}$:

$$\varphi_t = a_{20}[k(1+\epsilon)]\varphi_{xx} + \cdots. \tag{4.44}$$

We now expand the linear coefficient a_{20} to first order in ϵ:

$$a_{20}(k+\epsilon k) = a_{20}(k) + \epsilon g(k) + \mathrm{o}(\epsilon^2). \tag{4.45}$$

Finally,

$$g(k) = k\frac{\mathrm{d}}{\mathrm{d}k}a_{20}(k). \tag{4.46}$$

4.5.2. A-type zig-zag nonlinear coefficients
We will relate the nonlinear coefficient g_1 to the propagation speed, $c(k)$, of the X_k A-type structure. We notice a particular solution of the phase equation:

$$\varphi(x,t) = \epsilon x + [a_{10}(k) + g_1(k)c^2(k)]t, \tag{4.47}$$

which is valid for any ϵ. The perturbation of X_k obtained with this particular phase, may be seen as the structure $X_{k(1+\epsilon)}$, provided that ϵ is a small parameter. The expression for φ implies that such a perturbation propagates with a velocity $c(k) - a_{10}\epsilon - g_1(k)\epsilon^2$, which must be identical to $c(k+\epsilon k)$ to first order in ϵ. We deduce from this:

$$\begin{aligned} a_{10}(k) &= k\frac{\mathrm{d}}{\mathrm{d}k}c(k), \\ g_1(k) &= \frac{k^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}k^2}c(k). \end{aligned} \tag{4.48}$$

4.5.3. Zig-zag nonlinear coefficients
Similar arguments yield for the type-B zig-zag phase equation:

$$\begin{aligned} g_1(k) &= k\frac{\mathrm{d}}{\mathrm{d}k}a_{02}(k), \\ g_2(k) &= \frac{1}{2}k\frac{\mathrm{d}}{\mathrm{d}k}a_{02}(k) \end{aligned} \tag{4.49}$$

and for the type-A zig-zag phase equation:

$$g(k) = \frac{1}{2}k\frac{\mathrm{d}}{\mathrm{d}k}c(k). \tag{4.50}$$

5. Conclusion

5.1. Dynamical systems

The destabilization of fixed points in dissipative dynamical systems (finite system of ordinary differential equations) may be schematically reduced to the three most current bifurcations. There are first two stationary bifurcations, corresponding to a change of sign of a real eigenvalue of the linear stability operator: the saddle-node bifurcation in the generic case and the pitchfork bifurcation when the fixed point owns a symmetry of the equations. The last one is the oscillating destabilization when two complex conjugated eigenvalues cross the imaginary axis, known as the Hopf bifurcation.

The pitchfork bifurcation and the Hopf bifurcation are contained in their normal forms, which read, respectively, at the lowest order:

$$\begin{aligned}&\text{the Landau equation:} \quad \dot{W} = \mu W - \alpha|W|^2 W, \qquad \text{with} \quad \alpha \in \boldsymbol{R},\\ &\text{the Landau–Ginzburg equation:}\\ &\qquad \dot{W} = (\mu + \mathrm{i}\omega)W - \alpha|W|^2 W, \qquad \text{with} \quad \alpha \in \boldsymbol{C},\end{aligned} \tag{5.1}$$

where μ and $\mu + \mathrm{i}\omega$ are the crossing eigenvalues and $W(t)$ the amplitude of the marginal mode. The nonlinear coefficient α can be calculated, e.g., using the KBM method.

The asymptotic regimes of these instabilities are rather dull: new equilibria arise (fixed points) or sustained oscillations (limit cycle). The picture changes when more than one parameter is varied, since more than just one or two complex conjugated eigenvalues may then cross simultaneously the real axis on a surface in parameter space. Such a surface has a codimension higher than one. Rich dynamics may be found at the onset of instability for such bifurcations, and their normal forms are good models for the understanding of universal behavior (period doubling, Sil' Nikov orbits,etc.). These bifurcations may explain the dynamics observed far from the high-codimension surface in parameter space. But this carries us beyond the subject we are dealing with here.

5.2. Spatially confined systems

The destabilization of spatial hydrodynamic-like systems may be reduced to one of the dissipative dynamical systems if the spatial extension is confined. Indeed, the solutions may be decomposed on a discrete basis of modes. But since the equations are real, while the modes may be complex, and due to the symmetries of the equations and the studied equilibrium, the three previously mentioned bifurcations are not sufficient, in general, to describe the most current cases.

This approach is valid only if the marginal mode is homogeneous ($k_\mathrm{c} = 0$ for a Fourier decomposition). However, in the nonhomogeneous case (k_c finite) we must consider two coupled Landau–Ginzburg equations for the oscillating bifurcation. Small-box dynamics then consists in equilibria, propagating or standing nonlinear waves. Even this situation is not particularly enticing.

5.3. Spatially extended systems

Only with a sufficient spatial extension of the system does one encounter rich dynamics at the onset of a current bifurcation. In this large-box case it is as if continua of modes are becoming marginal, instead of discrete sets. The marginal continuum may be viewed as a wave packet, and one can describe the state of the system by replacing the amplitude $W(t)$ of the most unstable mode by an amplitude $W(x,t)$, slowly modulated in space. This complex function obeys amplitude equations which have the following typical shape:

the Landau equation:

$$W_t = \mu W + \beta\,\Delta W - \alpha|W|^2 W, \qquad \text{with} \quad \alpha, \beta \in \boldsymbol{R}, \tag{5.2}$$

the Landau–Ginzburg equation:

$$W_t = (\mu + \mathrm{i}\omega)W + \beta\,\Delta W - \alpha|W|^2 W, \qquad \text{with} \quad \alpha, \beta \in \boldsymbol{C}.$$

The Laplace operator, Δ, here comes from the Taylor expansion of the branch of marginal modes around its extremum. Other spatial operators may be found, such as $(\mathrm{d}x - \mathrm{i}\mathrm{d}y^2)^2$ for Rayleigh–Bénard convection.

The destabilized equilibrium state is then replaced by a continuum of solutions, instead of a discrete set as in the confined case. In the periodic case these continua of solutions may be A-type or B-type structures. The important fact is that these new structures may be unstable close to the bifurcation. The most frequent cause of instability is the existence of marginal modes due to the symmetries of the physical system. It is the study of these destabilizations which gave birth to the phase equation theory: the perturbed dynamics of these structures is reduced to the dynamics of one or several phases, each phase being associated with a particular symmetry of the physical system. The bifurcation parameter of these phase instabilities may be seen as the size of the spatial domain. The elimination of the damped modes allows to write phase equations describing the dynamics of the instability.

5.4. Supercritical or subcritical instabilities

We want to conclude this overview of normal forms, amplitude equations and phase equations by a general remark. It is important to check whether the studied instability is subcritical or supercritical. In the supercritical cases the reduction of the dynamics to a simple system (normal form, amplitude equation or phase equation) helps to get information on the new stable states reached after the destabilization: symmetric equilibria, A-type structures, phase chaotic motions, etc. But in the subcritical cases these simple systems blow up and are no longer valid when the perturbations are becoming too strong. This is the case for the Eckhaus instability, where nucleation of the B-type structures break the slavery of the amplitude modes. We can say that local analysis is possible when the destabilizations are supercritical, while global study of the system is needed in the subcritical cases.

However, these simple systems can be useful to build models, even for the subcritical cases. In the case of the Eckhaus instability, the theory of defects may be developed from simple models [11]. Models of subcritical destabilizations may be built by adding the next nonlinear term of the Landau or Landau–Ginzburg equation: $|W|^4W$. Investigations of these models may give information on the spatial coexistence of several states in the oscillating case, near the onset of the destabilization [12].

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